Australian Journal of Basic and Applied Sciences, 1(2): 109-114, 2007
ISSN 1991-8178

Using Feasible Directions to Solve Linear Fractional Programming Problems

S.F. Tantawy

Department of Mathematics, Faculty of Science, Helwan University (11795) Cairo, Egypt.

Abstract: In this paper, we give an iterative method for solving the problem when the objective function is linear fraction objectives and the constraints are in the form of inequality constraints. The proposed method based on conjugate gradient projection method which is applied to solve nonlinear programming problems with linear constraints. The main idea behind our method is to move through the feasible region via a sequence of points in the direction that improves the objective function. Since methods based on vertex information may have difficulties as the problem size increases this method may prove to be less sensitive to problem size. A simple example is given to clarify this solution procedure

Key words: linear fractional programming, conjugate gradient projection

INTRODUCTION

Linear fraction maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning.. Several methods to solve this problem are proposed in (1962), Charnes and Kooper have proposed their method depends on transforming this (LFP) to an equivalent linear program. Another method is called updated objective function method derived from Bit ran and Novas' (1973) is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function.Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by Bit ran and Magnant I (1976) and Singh.C. (1981) in his paper made a useful study about the optimality condition in fractional programming.

In this paper, we give an iterative method for solving the problem when the objective function is linear fraction objectives and the constraints are in the form of inequality constraints. The proposed method based on conjugate gradient projection method which is applied to solve nonlinear programming problems with linear constraints developed by Goldfarb, D.and Lapiduo, L. (1968). The main idea behind our method is to move through the feasible region via a sequence of points in the direction that improves the objective function. Since methods based on vertex information may have difficulties as the problem size increases this method may prove to be less sensitive to problem size. In section 2, we give full description of the problem together with our main results while section 3 contains the steps of our new algorithm with an example to illustrate how our algorithm works and finally section 4 constraints the main conclusion of this solution procedure

Definition and Theorems:

This mathematical programming problem arises when a linear fraction function is to be maximized on a convex constraint polyhedron X. this problem can be formulated as follows:

$$\text{Maximize} \quad F(x)=\frac{c^T x+\gamma}{d^T x+\beta} \tag{2-1}$$

Subject to
$x \in X = \{x,\ Ax \le b\}$

where $x \in R^n$, A is an $(m + n) \times n$ matrix, we point out that the nonnegative conditions are included in the set of constraints. Also c and d are n-vectors, $b \in R^{m+n}$ and γ,β. It is assumed that the feasible solution set X is a compact set i. e. bounded and closed. Moreover, $d^T x + \beta \succ 0$ everywhere in X.

Corresponding Author: S.F. Tantawy, Department of Mathematics, Faculty of Science, Helwan University (11795) Cairo, Egypt.
E-mail:saidtantawy@hotmail.com

This problem can also be formulated as

$$\text{Maximize} \quad F(x) = \frac{c^T x + \gamma}{d^T x + \beta}$$

Subject to

$$a_i^T x \le b_i \qquad i = 1, 2..., m + n. \qquad (2\text{-}2)$$

Here a_i^T represents the i th row of the given matrix A, then we have in the non degenerate case an extreme point (vertex) of X lies on some n linearly independent subset of X. We shall give an iterative method for solving this problem and our task is to find the optimal extreme point for this program, this method starts with an initial feasible point then a sequence of feasible directions toward optimality is generated to find the optimal extreme of this programming, in general if x^{k-1} is a feasible point obtained at iteration k-1 (k = 1, 2 ...) then at iteration k our procedure finds a new feasible point x^k given by

$$x^k = x^{k-1} + \alpha_{k-1} d^{k-1} \qquad (2\text{-}3)$$

Where d^{k-1} is the direction vector along which we move and given by

$$d^{k-1} = H_{k-1} \theta_{k-1} \qquad (2\text{-}4)$$

Here H_{k-1} is an n x n symmetric matrix given by

$$H_{k-1} = \begin{cases} I, & \text{for } k = 1 \\ H^q_{k-1} & \text{If } k > 1 \end{cases} \qquad (2\text{-}5)$$

In (2.5) we have I is an nxn identity matrix and q is the number of active constraints at the current point while

H^q_{k-1} is defined as follows, for each active constraint s;
s = 1, 2…,q. we have

$$H^s_{k-1} = H^{s-1}_{k-1} - \frac{H^{s-1}_{k-1} a_s a_s^T H^{s-1}_{k-1}}{a_s^T H^{s-1}_{k-1} a_s} \qquad (2\text{-}6)$$

With H^0_{k-1} = I.Then H_{k-1} is given by $H_{k-1} = H^q_{k-1}$ The step length α_{k-1} is given by

$$\alpha_{k-1} = \min_{i=1,\dots,m+n} \left\{ g_i \,/\, g_i = \frac{b_i - a_i^T x^{k-1}}{a_i^T d^{k-1}}, \text{ and } g_i > 0 \right\} \qquad (2\text{-}7)$$

This relation states that α_{k-1} is always positive. Proposition 2-2 below shows that such a positive value must exist if a feasible point exists. And θ_{k-1} is computed at the given point xk-1 to represent the local gradient at this point in the form,

$$\theta k\text{-}1 = (dT\ xk\text{-}1 + \beta)\ c - (cT\ xk\text{-}1 + \gamma)d \qquad (2\text{-}8)$$

Now consider the linear program

Maximize $F^*(x) = \theta_k^{\,T} x$

Subject to
$x \in X = \{x, Ax \le b\}$ (2-9)

With $\theta k = (d^T x^k + \beta) c - (c^T x^k + \gamma)d$, then we have the following proposition

Proposition 2-1:
If x^k solves the linear fractional programming (2-1) with optimal value $F(x^k)$ then xk solves the linear programming defined by (2-9) with optimal value

$F^*(x^k) = (\beta F(x^k) - \gamma)(d^T x^k + \beta)$.

Proof: straight forward

Due to the well known Kuhn-Takucer condition [6], foe xk to be an optimal solution of the linear program (2-9) there must exist $u \geq 0$ such that $A^T u = \theta k$, or simply $u = (ArA\ T)^{-1}Ar\ \theta k$ (2-10)
Here A_r is a submatrix of the given matrix A containing only the coefficients of the set of active constraints at the current point x^k. This fact will act as a stopping rule of our proposed algorithm, also we have to point out that the matrix $(H_{k-1})^2 = H_{k-1,}$ through the following proposition

Proposition 2-2:
For H_{k-1} defined by relation (2.5) above we have $(H_{k-1})^2 = H_{k-1}$.

Proof:
This can be proved by induction, define a matrix $Q_1 = \frac{a_1 a_1^T}{a_1^T a_1}$ and since $H^1_{k-1} = (I - \frac{a_1 a_1^T}{a_1^T a_1})$ then

$H^1_{k-1} Q_1 = 0, Q_1^2 = Q_1, (H_{k-1})^2 = H^1_{k-1}$ and H^1_{k-1} is an orthongal projective matrix. Also, if we define

$Q_2 = \frac{a_2 a_2^T H^1_{k-1}}{a_2^T H^1_{k-1} a_2}$ and $H^*_{k-1} = (I - \frac{a_2 a_2^T H^1_{k-1}}{a_2^T H^1_{k-1} a_2})$ then since $H^2_{k-1} = H^1_{k-1}(I - \frac{a_2 a_2^T H^1_{k-1}}{a_2^T H^1_{k-1} a_2})$ we have

$H^*_{k-1} Q_2 = 0, Q_2^2 = Q_2$ and $(H^*_{k-1})^2\ H^*_{k-1}$ Now, since $H^2_{k-1} = H^1_{k-1} H^*_{k-1}$ and both matrices H^1_{k-1} and H^*_{k-1} re

orthogonal projective, then

H^2_{k-1} is orthogonal projective matrix and we have

$(H^2_{k-1})^2 = H^2_{k-1}$ applying the same argument, we conclude that $H_{k-1} = H^r_{k-1}$ an orthogonal projective matrix

such that $(H_{k-1})^2 = H_{k-1}$.

Proposition 2-3:
Any solution x^k given by (2-3) is feasible and increases the objective function value.

Proof:

$$F(x^k) - F(x^{k-1}) = \frac{c^T x^k + \gamma}{d^T x^k + \beta} - \frac{c^T x^{k-1} + \gamma}{d^T x^{k-1} + \beta}$$
$$= \alpha_{k-1} \theta^T_{k-1} d^{k-1} / (d^T x^k + \beta)(d^T x^{k-1} + \beta),$$
$$= \alpha_{k-1} \theta^T_{k-1} H_{k-1} \theta_{k-1} / (dT\ xk + \beta)(dT\ xk-1 + \beta)$$
$$= \alpha_{k-1} \| H_{k-1} \theta_{k-1} \| / (d^T x^k + \beta)(d^T x^{k-1} + \beta) > 0$$

Since the denominator of this difference is positive. This proves that x^k increases the objective function. Next, we shall prove that x^k is a feasible point.
For x^k to be a feasible point it must satisfy all constraints of problem (2-1), then

$$a_i(x^{k-1} + \alpha_{k-1} d^{k-1}) \leq b_i$$

Must hold for all i Є {1, 2… m+n} which can be written

$$a_i \alpha_{k-1} d^{k-1} \leq b_i - a_i x^{k-1}, \quad i = 1, 2 \ldots, m+n$$

And this is valid for any i since if there is t∈ {1, 2… m+n} such that

$$a_t^T d^{k-1} > 0 \text{ and } a_t^T d^{k-1} > b_t - a_t^T x^{k-1}, \text{than } \frac{b_t - a_t^T x^{k-1}}{a_t^T d^{k-1}} < \alpha_{k-1}$$

That will contradict our definition of α_{k-1}. Next, we shall give a result that guarantees the existence of α_{k-1} defined by relation (2-7) above.

Proposition 2-3:

At any iteration k if a feasible point that will increase the objective function exists then α_{k-1} as defined by relation (2-7) must exist.

Proof:

To prove this result it is enough to prove that

$$a_i^T d^{k-1} \leq 0$$

Can not be true for all i ∈ {1,2,…, m+n}.Now suppose that relation (2-9) is true for i ∈ {1,2…, m+n} then writing (2-9) in matrix form and multiplying both sides by

$$u^T \geq 0, \text{we get} \quad u^T A d^{k-1} \leq 0$$

$$\text{i.e.,} \quad u^T A H_{k-1} \theta_{k-1} \leq 0$$

Since $u^T A = \theta^T_{k-1,}$ we have $\| H_{k-1} \theta_{k-1} \| \leq 0$, This contradicts the fact that the norm must be positive, which implies that relation (2-7) cannot be true for all i ∈ {1, 2… m+n}. Thus if a feasible point x^k exists then α_{k-1} as defined by relation (2-7) must exist. Based on the above results we shall give in the next section a full description of our algorithm for solving the linear fraction programming problem

3- New Algorithm for Solving the Linear Fraction Programming:

Our algorithm for solving the (LFP) problem consists of the following steps

Step 0: set k=1, Ho =I, $d^0 = \theta_0$, let x^0 be an initial feasible point and use relation (2-7) to compute α_0.

Step 1: Apply relation (2-3) to find a new solution x^k.

Step 2: Apply relation (2-10) to compute u, if u ≥ 0 stop. The current solution x^k is the optimal solution otherwise go to step 3.

Step 3: Set k = k+1, apply relations (2-6), (2-4) and (2-7) to compute $H_{k-1,}$ $d^{k-1}$and α_{k-1} respectively and go to step 1.

Given an initial feasible point x^0 and a vector c, step 0 computes α_0 in O(m+n) steps. Computing x^k in step 1 requires O(n) steps while testing the optimality of the current solution x^k, in step 2 requires $O(n^3)$ steps. Step 3 of the algorithm requires $O(n^3)$ steps to compute H_{k-1} while computing d^{k-1}, the feasible direction that increase the value of the objective function, requires $O(n^2)$ steps, finally to compute α_{k-1} requires O(m+n) steps. Hence the application of each iteration of our algorithm requires $O(\max\{m+n, n^3\})$ steps. Proposition 3-1 below states that the above algorithm solves the above problem in at most m+n iteration.

Remark 3-1:

Assuming that q is the number of active constraints at point x^k then if q<n and relation (2-9) is satisfied this indicates that x^k is an optimal non-extreme point, in this case the objective function can not be improved through any feasible direction and we have $H_k \theta_{k.} = 0$ at this point x^k, we note that although the matrix H_k is singular in all iterations in this case we have all subsequent search directions d^{k+1} will be orthogonal to $\theta_{k.}$

Proposition 3-1:

Our algorithm solves the mathematical programming problem given by (2-1) in at most m+n iterations.

Proof:

For this algorithm at least one constraint is added at a time starting with $H_0^0 = I$ then an optimal extreme point may be reached in n steps and the algorithm terminate in at most n iterations. On the other hand if at a given iteration we have non optimal extreme point and at least one constraint has to be dropped from the set of active constraints, this constraint can not be active again at any subsequent iterations of the algorithm. Since our allowed directions (given by 2-4) that improve the value of the objective function lies in the nullity of a subset of the given matrix A, then we are moving in the direction parallel to a certain subset of the (m+n) constraints and hence in the worst case the maximum number of iterations required to reach the optimal point is limited by m+n.

An Illustrative Example:

Example: Consider the following (LFP) problem

Maximize $F(x) = \dfrac{x_2+1}{x_1+3}$

Subject to:

$$
\begin{aligned}
- x_1 + x_2 &\le 1\\
x_2 &\le 2\\
x_1 + 2x_2 &\le 7\\
x_1 &\le 5\\
- x_1 \le 0,\ - x_2 &\le 0
\end{aligned}
$$

To solve this (LFP) problem by our suggested procedure we have to go through the following steps,

Step 0: k=1, $H_{0=}$, $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $\theta^0 = d_0 = \begin{pmatrix}-2\\4\end{pmatrix}$

Let $x^0 = \begin{pmatrix}1\\1\end{pmatrix}$ ıe an initial feasible point, then (2-7) gives α_0 =1/6 and we go to step 1

Step 1: apply relation (2-3) to get $x^1 = \begin{pmatrix}1\\1\end{pmatrix} - 1/6\begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}2/3\\5/3\end{pmatrix}$, ɘ got to step 2

Step 2: for this point x^1 the first constraint is the only active constraint and since relation (2-8) is not satisfied indicates this point is not optimal.

Step 3: set k=2

$$H_1\begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix},\ d^1 = \begin{pmatrix}1/2\\1/2\end{pmatrix}$$

and α_1 =2/3and we go to step 1.

Step 1: apply relation (2-3) to get $x^2 = \begin{pmatrix}2/3\\5/3\end{pmatrix} + 2/3\begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

Since x^2 satisfies the first two constraints as equalities we go to step, apply relation to get $\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$,

hence this indicates that the point $x^2 = \begin{pmatrix}1\\2\end{pmatrix}$; the optimal solution for this linear fraction programming problem with optimal

value F(x) = 3/4

Conclusion:

In this paper we gave an iterative procedure for solving the (LFP) problem when the objective function is linear fraction and the constraints of the form of linear inequality constraints.Our procedure is based on modifying the conjugate gradient method for solving nonlinear programming problems with linear constraints to handle this mathematical programming problem Starting with an initial feasible point then a sequence of feasible directions toward optimality is generated. Our new procedure can be applied to linear programming problems since they are special cases of this mathematical program

REFERENCES

Bit ran, G.R. and A.J. Novaes, 1973. Lin ear programming wit h a fractional objective function. Operat ions Research, vol 21, No 4, pp: 22-29.

Bit ran, G.R. and T.L. Magnant, 1976. Duality and sensitivity analysis with fract ional objective funct ion. Operation Research, vol 24, pp: 675-699.

Chaners, A. and W.W. Cooper, 1962. Programming wit h linear fractional functionals. Nava l Research log istics Quarterly, vo l 9, No 3-4, pp: 181-186.

Goldfarb, D., 1969. Extension of Davidson's variable metric method to maximization under linear inequality and equality constraints, SIAM J. Appl. Math., 17: 739-764.

Goldfarb, D. and L. Lapiduo, 1968. Conjugate gradient method for non linear programming problems with linear constraints, Ind & eng. Chem. Fund, 7: 148-151.

Greig, D.M., 1980. Optimization.Longman.london and New York.

Sing, H.C., 1981. Optimality condition in factional programming. Journal of Optimization Theory and Applications, vol 3 3, pp: 287-294.