Australian Journal of Basic and Applied Sciences, 2(4): 1378-1380, 2008
ISSN 1991-8178

On Asymptotic Distribution of Sample Variance In Skew Normal Distribution

Narges Abbasi

Department of Statistics, Payame Noor University,
Shiraz Centre 71365-944, Shiraz, Iran.

Abstract: The univariate skew normal distribution was introduced by Azzalini(1985). The properties of the resulting distribution were studied. The main aim of the paper is to find the limiting distribution of sample variance in skew normal distribution. In this way, the theory of U-statistics is applied.

Key words: Mmoments, Limiting distribution, Skew normal distribution, Symmetric Kernel, U-Statistics.

INTRODUCTION

The univariate skew normal distribution has been considered by several authors. Azzalini (1985) introduced this distribution. A random variable X has skew normal distribution if

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi(\lambda x), \qquad x \in R$$

where Φ is cumulative distribution function of standard normal distribution. The skew normal distribution allows for continuous variation for continuous variation from normality to non-normality, which is useful in many practical situations (Hill and Dixon, 1982, Arnold et al., 1983). Nadarajah, et al., (2003) generated skewed probability density function of the form $2f(u)G(\lambda u)$ where f is taken to be a normal pdf while the cumulative distributive function G is taken to come from one of normal, Student's t, Cauchy, Laplace, logistic or uniform distribution. In particular, expressions for the nth moment and characteristic function were derived. In skew normal distribution, the nth moment of X about zero turns out to be

$$E(X^n) = \frac{2^{\frac{n+2}{2}} \lambda}{\pi} \Gamma\left(\frac{n+2}{2}\right) \left(1+\lambda^2\right)^{\frac{-n}{2}} \sum_{k=0}^{\frac{n-1}{2}} \frac{\left(\frac{1-n}{2}\right)_k}{\left(\frac{3}{2}\right)_k} \left(-\lambda^2\right)^k, \qquad (1)$$

if n is odd, and

$$E(X^n) = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right), \qquad (2)$$

for n even, where $(c)_k = c(c+1)\cdots(c+k-1)$

In this paper, by using these results, we derive the asymptotic distribution of sample variance in skew normal distribution.

Corresponding Author: Narges Abbasi, Department of Statistics, Payame Noor University, Shiraz Centre 71365-944, Shiraz, Iran.
E-mail: abbasi@spnu.ac.ir

RESULT AND DISCUSSION

Let $X_1, X_2, \ldots, X_n$ be a random sample from skew normal distribution. The U-statistic estimator of the variance with the symmetric kernel $h(X_1, X_2) = \frac{1}{2}(X_1 - X_2)^2$ s

$$
\begin{aligned}
U_2(X_1, \ldots, X_n) &= \frac{1}{\binom{n}{2}} \sum_{\beta \in B} \frac{1}{2}\left[\sum_{i=1}^{2} X_{\beta_i}^2 - 2X_{\beta_1} X_{\beta_2}\right] \\
&= \frac{1}{n(n-1)}\left[n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= S^2,
\end{aligned}
$$

where $B = \{\beta \mid \beta$ is one of the $\binom{n}{2}$ unordered subsets of 2 integers chosen without replacement from the set $\{1,2,\ldots,n\}\}$ (Koroljik and Borovskich 1994). Since the conditions of theorem 3 from Ferguson (2005), ($\mu_4 = E(X-\mu)^4 \; \sigma^4 < \mu_4 < \infty$, $E_F h^2 < \infty$) holds, we conclude that the sample variance has a limiting normal distribution with mean σ^2 and varia $\frac{\mu_4 - \sigma^4}{n}$

We will employ (1) and (2) to establish the limiting distribution of S^2.

$$
E(X) = \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{1}{\sqrt{1+\lambda^2}}, \qquad E(X^2) = 1,
$$

$$
E(X^3) = \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{3+2\lambda^2}{\left(1+\lambda^2\right)^{\frac{3}{2}}}, \qquad E(X^4) = 3.
$$

Therefore

$$\mu_4 - \sigma^4 = E(X^4) - 4E(X^3)E(X) + 8E(X^2)E^2(X) - E^2(X^2) - 4E^4(X)$$

$$= 3 - 4\frac{\sqrt{2}}{\sqrt{\pi}}\lambda\frac{3+2\lambda^2}{(1+\lambda^2)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{\pi}}\lambda\frac{1}{\sqrt{1+\lambda^2}} + 8\left(\frac{\sqrt{2}}{\sqrt{\pi}}\lambda\frac{1}{\sqrt{1+\lambda^2}}\right)^2$$

$$-1 - 4\left(\frac{\sqrt{2}}{\sqrt{\pi}}\lambda\frac{1}{\sqrt{1+\lambda^2}}\right)^4$$

$$= 2 - \frac{8\lambda^2}{\pi}\frac{3+2\lambda^2}{(1+\lambda^2)^2} + \frac{16\lambda^2}{\pi}\frac{1}{(1+\lambda^2)} - \frac{16\lambda^2}{\pi^2}\frac{1}{(1+\lambda^2)^2}$$

$$= 2 - \frac{8\lambda^2}{\pi^2(1+\lambda^2)^2}[3\pi + 2\pi\lambda^2 - 2\pi - 2\pi\lambda^2 + 2\lambda^2]$$

$$= 2 - \frac{8\lambda^2(\pi + 2\lambda^2)}{\pi^2(1+\lambda^2)^2}.$$

So, $\sqrt{n}(S^2 - 1 + \frac{2\lambda^2}{\pi(1+\lambda^2)}) \sim AN(0, 2 - \frac{8\lambda^2(\pi+2\lambda^2)}{\pi^2(1+\lambda^2)^2})$. If λ tends to infinity the above value near to $2 - \frac{16}{\pi^2}$ and

$$\sqrt{n}(S^2 - 1 + \frac{2}{\pi}) \sim AN(0, 2 - \frac{16}{\pi^2}).$$

REFERENCES

Arnold, B.C., R.J. Beaver, 2000. Some skewed multivariate distributions. Amer. J. Math. Management Sci., 20: 27-38.

Azzalini, A., 1985. A class of distributions which includes the normal ones. Scand. J. Statist., 12: 171-178.

Ferguson, T.S., 2005. U-Statistics, Notes for statistics 200C, spring. www.math.ucla.edu/~tom/Stat200C/Ustat.pdf.

Hill, M.A., W.J. Dixon, 1982. Robustness in real life: a study of clinical laboratory data. Biometrics, 38: 377-396.

Koroljik, V.S. and Yu.V. Borovskich, 1994. Theory of U-Statistics. Kluwer Academic Publishers, Dordrecht-Boston-London.

Lee, A.J., 1990. U-statistics: Theory and Practice, New York: Dekker.

Nadarajah, S. and S. Kotz, 2003. Skewed Distributions Generated by the Normal Kernal, Statistics & Probability Letters, 65: 269-277.