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A Statistical Approach to the Study of Qualitative Behavior of Solutions of Second Order Neutral Differential Equations

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Abstract: In this paper, we have given a statistical approach to the study of qualitative behavior of solution of a second order neutral differential equations of the form $y^{//}(t) = p_1y^{/}(t) + p_2\ y^{/}(t-\tau) + q_1y(t) + q_2y(t-\tau) + r_1\ y^{//}(t-\tau)$, $t \geq 0$ with the initial condition $y(t)= \varphi(t)$, $-\tau \leq t \leq 0$ where p_1, p_2, q_1, q_2, and r_1 and τ are real constants and $t > 0$. The idea is to merge research in computer experiments with that in differential equations and use the same to get much valuable information on how parameters (here the real constants defined above) and their interactions affect the solution. We strongly emphasize that the information on the interactive effects are quite an imbroglio from a theoretical angle. The strength of such an untraditional approach cannot be thrown away.

Key words: 2000 AMS Subject classification: 34 K 15, 62 K 15, neutral differential equation; computer experiments; statistical approach.

INTRODUCTION

In this paper, an attempt has been made to study the qualitative behavior of solution of the second order neutral differential equations of the form

$$y^{//}(t) = p_1y^{/}(t) + p_2\ y^{/}(t-\tau) + q_1y(t) + q_2y(t-\tau) + r_1\ y^{//}(t-\tau),\ t \geq 0 \quad (1)$$

with the initial condition
$y(t)=\varphi(t)$, $-t \leq t \leq 0$

where p_1, p_2, q_1, q_2, and r_1 and τ are real constants and $\tau > 0$ using factorial experiments.
Equation of the form (1) has been studied theoretically by many authors on the oscillation and non-oscillation, stability and asymptotic stability of the solution of the equation. For instance, see (A.F. Yenicerioglu, 2006; I. Gyori and G. Ladas, 1991) and the references cited therein.
Setting $y(t) = e^{\lambda t}$, we obtain the following characteristic equation of (1)

$$\tau^2 = p_1\lambda + p_2\ \lambda e^{-\lambda t} + q_1 + q_2\ e^{-\lambda t} + r_1\lambda^2 e^{-\lambda\tau} \quad (2)$$

Set $F(\lambda) = -\lambda^2 + p_1\lambda + p_2\ e^{-\lambda t} + q_1 + q_2\ e^{-\lambda\tau} + r_1\lambda^2 e^{-\lambda\tau}$

We have made a computational study of the characteristic equation (2) to analyze the behavior of solution of (1) and have studied the same for simultaneous variation of the associated parameters in (2). In a recent work (A. Kumari and S. Chakraborty, 2007), factorial experiments have been successfully used in the context of computer experiments. The motivation for the present work on computer experiments has come from the work of Sacks *et. al.* (1989).

Experimental Findings by Programming Newton-raphson Method to Solve (2):

Our purpose is to study the behavior of the solution of the neutral differential equation for *simultaneous variation of the associated* parameters p_1, p_2, q_1, $q_{2,}$ t and r_1 each within certain limit. Using the concept of *factorial experiments* as used in statistics we shall, taking the parameters as treatments, study which main effects and interaction effects are significant in a comparative sense. For this purpose we shall conduct computer experiments.

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A computer experiment is a series of runs of a code for various inputs. A deterministic computer experiment is one in which re-running the code with identical inputs leads to identical responses. Most computer experiments are deterministic and this is true irrespective of whether the response is the output or a complexity (A. Kumari and S. Chakraborty, 1989). However by realizing the response as the outcome of a stochastic process it is possible to derive some potential benefits such as reducing the cost of prediction (this means predicting the response for untried inputs for which it may be computationally cumbersome to run the code). This was in fact the motivation of Sacks and others in ref. (J. Sacks, *et al.,* 1989). Here also the situation is deterministic as the root is strictly determinable for a given set of parameters and fixed t so that if we run the code for the same set of parameters invariably we shall get identical results. Following the principle advocated by Sacks (1989) we are motivated to *realize the deterministic response of our neutral differential equation as the outcome of a stochastic process and conduct factorial experiments to obtain much valuable information on the behavior of the solution with special emphasis on both the main treatments (parameters) and their interaction effects.* However four different values of tau are used to generate some "noise". This helps us in conducting the factorial experiment with four blocks (replicates). The three degrees of freedom arising from the four blocks are added to the error degrees of freedom so that treatments can be compared with greater precision. This is precisely what is done in the MINITAB statistical package where we have analyzed the results of the computer experiments.

Experimental study 1 (2^5 Factorial experiment):

We shall conduct a 2^5 factorial experiment (five factors each at two levels).

Using standard notation, in table 1.1 to 1.4 the parameters p_1 ,p_2 , q_1 , q_2 and r_1 (which we shall denote as p1, p2, q1, q2, r1) correspond to a, b, c, d, e respectively. Levels are 1 and 10 for each. Presence of the alphabet (a-e) indicates the corresponding parameter (treatment) is at the second level while absence indicates it is at first level. Table 2 gives MINITAB Results for 2^5 factorial experiment.

Table 1.1: Block 1 Observations of root for τ =1

Treatment combination	observation	Treatment combination	observation
1	1.983816	a	-0.1986127
b	2.480973	ab	-0.1000501
c	3.769217	ac	10.91628
bc	3.874468	abc	-0.4541431
d	2.252577	ad	10.09951
bd	2.602551	abd	-0.7244475
cd	3.798828	acd	10.9163
bcd	3.898674	abcd	-1.053626
e	2.951403	ae	10.10316
be	-.9254288	abe	-0.1053113
ce	4.118416	ace	10.91792
bce	4.183823	abce	-0.7939879
de	3.018016	ade	10.1032
bde	3.181357	abde	9.120828
cde	4.134904	acde	10.91794
bcde	4.198407	abcde	10.91809

Table 1.2: Block 2, Observations of root for t =1.5;

Treatment combination	observation	Treatment combination	observation
1	1.788203	a	-0.2098762
b	2.058192	ab	-0.1001137
c	3.712563	ac	10.91608
bc	3.731666	abc	-0.4185706
d	1.94817	ad	10.09902
bd	2.142543	abd	-0.8408478
cd	3.717804	acd	10.91608
bcd	3.736622	abcd	-1.077648
e	2.256607	ae	10.09905
be	2.381564	abe	-1.104142
ce	3.780103	ace	10.91609
bce	3.796355	abce	-0.8658726
de	2.315285	ade	10.09905
bde	2.423905	abde	10.09905
cde	3.78447	acde	10.91609
bcde	3.800517	abcde	10.91609

Table 1.3: Block 3 Observations of root for t =2;

Treatment combination	observation	Treatment combination	observation
1	1.698688	a	-0.2243295
b	1.848349	ab	-0.1001001
c	3.703315	ac	10.91608
bc	3.70644	abc	-0.389009
d	1.791177	ad	9.109773
bd	1.904938	abd	-0.9494605
cd	3.704157	acd	9.908327
bcd	3.707282	abcd	-1.096045
e	1.943836	ae	10.09902
be	2.02708	abe	-1.030987
ce	3.714762	ace	10.91608
bce	3.717772	abce	-0.8768924
de	1.988531	ade	10.09902
bde	2.060608	abde	10.09902
cde	3.715575	acde	10.91608
bcde	3.718576	abcde	10.91608

Table 1.4: Block 4 Observations of root for t =2.5;

Treatment combination	observation	Treatment combination	observation
1	1.655846	a	-0.2441665
b	1.736315	ab	-0.100125
c	3.701845	ac	10.91608
bc	3.702334	abc	-0.3639707
d	1.706333	ad	10.09902
bd	1.771908	abd	-1.02592
cd	3.701972	acd	10.91608
bcd	3.702479	abcd	-1.108462
e	1.784485	ae	10.09902
be	1.837102	abe	-0.9835972
ce	3.703672	ace	10.00000
bce	3.704171	abce	-0.8814302
de	1.814908	ade	9.109773
bde	1.861232	abde	10.09902
cde	3.703805	acde	10.91608
bcde	3.704297	abcde	10.9160

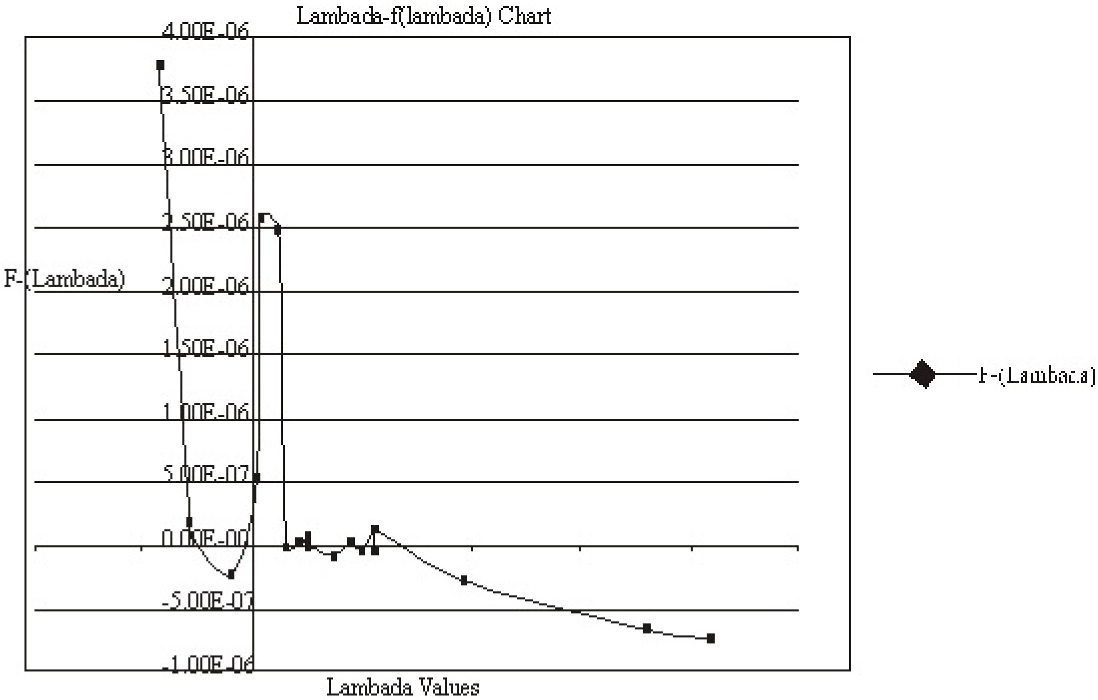


Fig. 1:

Table 2: MINITAB Results for 2^5 factorial experiment

General Linear Model: observation versus level a, level b, ...

Factor	Type Levels	Values
level a	fixed	2 0 1
level b	fixed	2 0 1
level c	fixed	2 0 1
level d	fixed	2 0 1
level e	fixed	2 0 1

Analysis of Variance for observant, using Adjusted SS for Tests

Source	DF	Seq SS	Adj SS	Adj MS	F	P
level a	1	266.149	266.149	266.149	238.15	0.000
level b	1	360.316	360.316	360.316	322.41	0.000
level c	1	81.808	81.808	81.808	73.20	0.000
level d	1	106.196	106.196	106.196	95.02	0.000
level e	1	115.207	115.207	115.207	103.09	0.000
level a*level b	1	347.374	347.374	347.374	310.83	0.000
level a*level c	1	0.792	0.792	0.792	0.71	0.402
level a*level d	1	87.948	87.948	87.948	78.70	0.000
level a*level e	1	100.457	100.457	100.457	89.89	0.000
level b*level c	1	29.647	29.647	29.647	26.53	0.000
level b*level d	1	10.424	10.424	10.424	9.33	0.003
level b*level e	1	7.268	7.268	7.268	6.50	0.012
level c*level d	1	6.811	6.811	6.811	6.09	0.015
level c*level e	1	3.943	3.943	3.943	3.53	0.063
level d*level e	1	38.484	38.484	38.484	34.44	0.000
level a*level b*level c	1	31.729	31.729	31.729	28.39	0.000
level a*level b*level d	1	6.734	6.734	6.734	6.03	0.016
level a*level b*level e	1	11.784	11.784	11.784	10.54	0.002
level a*level c*level d	1	2.793	2.793	2.793	2.50	0.117
level a*level c*level e	1	4.615	4.615	4.615	4.13	0.045
level a*level d*level e	1	30.889	30.889	30.889	27.64	0.000
level b*level c*level d	1	20.068	20.068	20.068	17.96	0.000
level b*level c*level e	1	32.722	32.722	32.722	29.28	0.000
level b*level d*level e	1	174.961	174.961	174.961	156.55	0.000
level c*level d*level e	1	8.030	8.030	8.030	7.19	0.009
level a*level b*level c* level d	1	26.685	26.685	26.685	23.88	0.000
level a*level b*level c* level e	1	22.169	22.169	22.169	19.84	0.000
level a*level b*level d* level e	1	154.886	154.886	154.886	138.59	0.000
level a*level c*level d* level e	1	11.825	11.825	11.825	10.58	0.002
level b*level c*level d* level e	1	23.679	23.679	23.679	21.19	0.000
level a*level b*level c* level d*level e	1	17.098	17.098	17.098	15.30	0.000
Error	96	107.287	107.287	1.118		
Total	127	2250.777				

Table 3:

λ	p1	p2	q1	q2	r1	τ	F(λ)
-0.20255	-1.28905	0.250426	0.475602	-0.62294	-0.52118	0.004431	-2.15E-07
0.492312	0.276235	-1.28671	0.166161	0.23284	1.265925	0.914925	9.68E-09
0.891099	-0.95341	-1.03886	1.38757	0.982977	0.524446	0.690306	2.84E-08
0.425427	-0.72048	-0.46671	-0.11571	0.902598	-0.54409	0.009194	3.11E-08
3.593622	-0.18615	1.045526	0.785614	2.19059	3.123575	0.357743	-6.46E-07
-0.58035	0.704021	-0.63067	-1.02791	-0.41334	1.14386	2.856523	1.99E-07
1.109802	0.441099	-1.84191	1.138491	-1.13107	-0.18152	1.936242	-1.67E-08
0.492061	0.785206	-0.87988	-0.36902	1.053011	0.093642	2.135134	7.92E-08
4.175102	0.204619	0.324631	1.224754	-0.74612	2.383588	0.241946	-7.23E-07
0.068479	-0.38971	-0.69793	-0.50496	0.747963	-0.74234	3.819876	2.59E-06
1.914102	1.584151	-0.54913	0.48407	0.133819	0.467542	0.880551	-2.67E-07
-0.56409	-1.58705	-2.70201	-0.01367	-1.7448	-0.27235	1.074511	8.63E-08
1.126863	-1.14519	0.834635	1.676731	1.18091	-0.49805	0.463137	1.20E-07
0.72726	0.228984	-1.40327	1.170452	0.135472	-0.05809	0.172067	-6.61E-08
0.987567	-1.37512	0.970744	2.63704	-2.04293	0.57769	0.546087	-2.58E-08
-0.84593	0.468704	-0.6082	-0.17038	-0.63294	1.202602	0.646631	3.77E-06
1.091765	-0.01976	0.253406	1.103686	0.222198	-0.16046	0.943285	1.36E-07
0.038459	-1.85743	-2.6164	-0.07192	0.244498	2.570682	0.506116	5.46E-07
0.293097	0.633068	-0.06579	-0.12607	0.110533	-0.5841	1.504855	1.10E-09
0.228102	0.309665	-1.62035	1.645236	-1.30173	-2.66063	0.368559	2.48E-06

In addition to the p-values in table 2, we also have the analysis for 5% level of significance:-

F(1, 80) at 5% level=3.96 while F(1, 100) at 5% level=3.94. Hence F(1, 95) at 5% level is some value lying in the interval [3.94, 3.96]. We did not feel it necessary to interpolate or regress to estimate this F value since none of the calculated values of F falls in this interval. All calculated F values are either below 3.94 or above 3.96 and hence clearly either insignificant or significant at 5% level of significance respectively. The general conclusions are as under:-

- Single Factor effects: All are highly significant
- Two factor effects: ac, ce are insignificant. bd, be, cd are significant but not highly.
- All others are highly significant.
- Three factor effects: acd is insignificant. abd, ace, cde are significant but not highly.
- All others are highly significant.
- Four factor interactions: acde is significant but not highly. Others are highly significant.
- Five Factor interaction: The sole five factor interaction abcde is significant certainly though not highly.

Experimental Study 2 (Determining a Region for Dominant Root):

Determining a region empirically by direct programming for those values of p_1, p_2, q_1, q_2, r_1 and τ for which λ is the dominant root. All the first five parameters are allowed to vary randomly as independent standard normal variates (Box Muller Transformation (W. Kennedy and J. Gentle, 1980) used for simulation). τ values are taken as independent exponential variate with mean 1(inverse cdf technique (W. Kennedy and J. Gentle, 1980) used for simulation). Values of the root for randomly selected parameters p1, p2, q1, q2, r1 and τ within their prescribed limit. See table 3.

The QBASIC program, which generated the data given in table 3, is as follows:

```
REM root of a diff equation by Newton Raphson method
CLS
    5   accu = .001
    RANDOMIZE TIMER
    tau = (-1) * LOG(1 - RND): PRINT "tau="; tau
    p1 = SQR((-2) * LOG(RND)) * COS(2 * 4 * ATN(1) * RND)
    p2 = SQR((-2) * LOG(RND)) * SIN(2 * 4 * ATN(1) * RND)
    q1 = SQR((-2) * LOG(RND)) * COS(2 * 4 * ATN(1) * RND)
    q2 = SQR((-2) * LOG(RND)) * SIN(2 * 4 * ATN(1) * RND)
    r1 = SQR((-2) * LOG(RND)) * COS(2 * 4 * ATN(1) * RND)
    l = 1'l is the initial guess'
    10  fl = (-1) * l * l + p1 * l + (r1 * l * l + p2 * l + q2) * EXP((-1) * l * tau) + q1
    fld = (-2) * l + p1 + (2 * r1 * l + p2) * EXP((-1) * l * tau) - tau * (r1 * l * l + p2 * l + q2) * EXP((-1)
    * l * tau)
    lnew = l - fl / fld
    IF ABS(l - lnew) >= accu THEN l = lnew: GOTO 10
    l = lnew
    fl = (-1) * l * l + p1 * l + (r1 * l * l + p2 * l + q2) * EXP((-1) * l * tau) + q1
    IF fl < .001 THEN PRINT "f(lambda)="; fl ELSE GOTO 5
    PRINT "root="; lnew
    PRINT "p1,p2,q1,q2,r1 are respectively"; p1, p2, q1, q2, r1
    END
```

Remark:

This program was modified for the factorial experiment case for four different values of t and fixing values of the remaining parameters, for each t value, according to the treatment combination.

Behavior of the Solution:

The data in table 3, which we generated through programming, has the advantage of getting the solution of the differential equation as well as the dominant root, directly by simulating all the parameters p1, p2, q1, q2, r1 and τ within their domain. We have observed the relations between (i) λ and $F(\lambda)$ denoted as Lambda and F(Lambda) respectively in the graph.

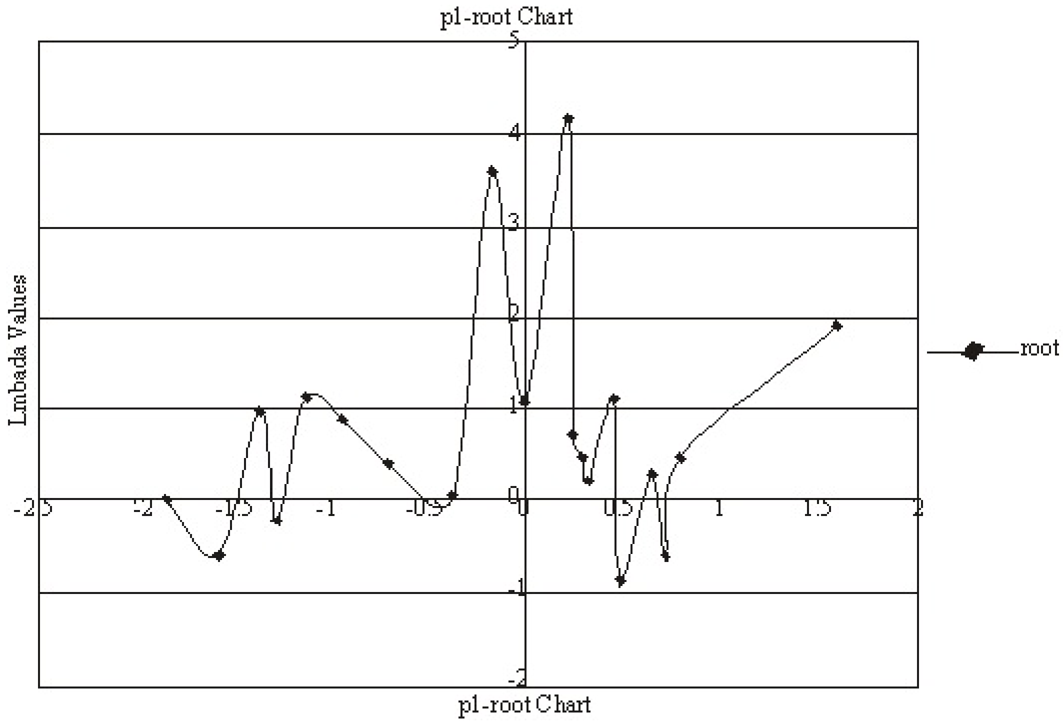

p1-root Chart
Lmbada Values
root
p1-root Chart

Fig. 2:

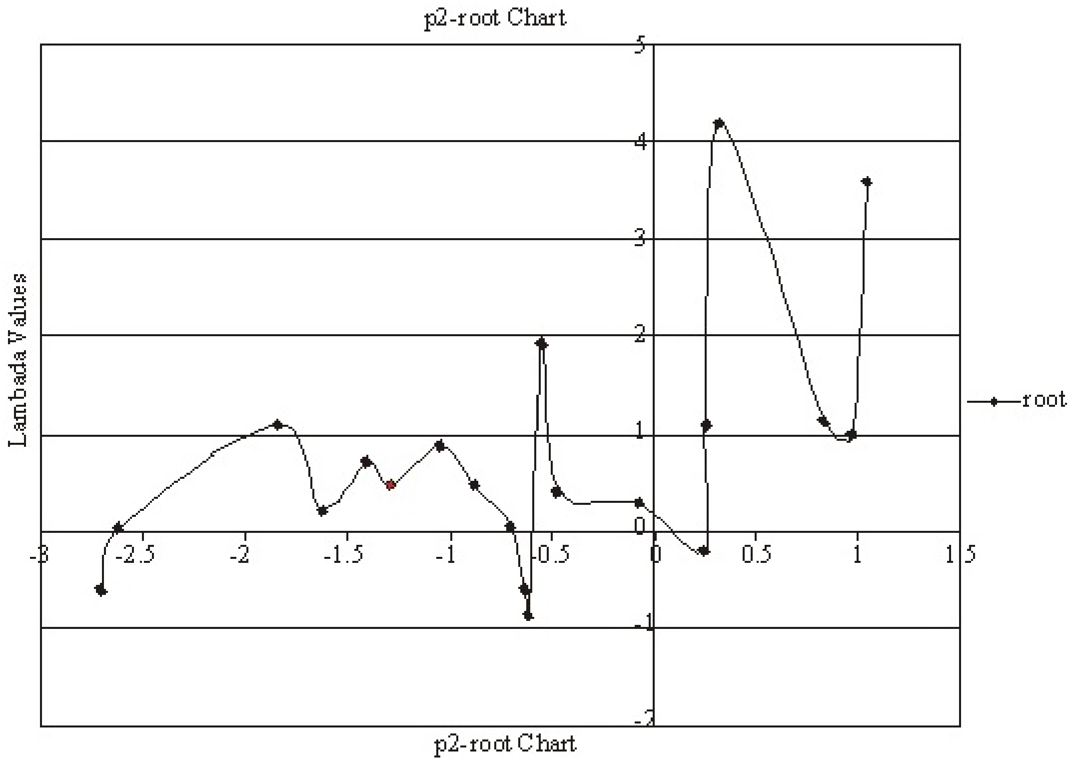

p2-root Chart
Lambada Values
root
p2-root Chart

Fig. 3:

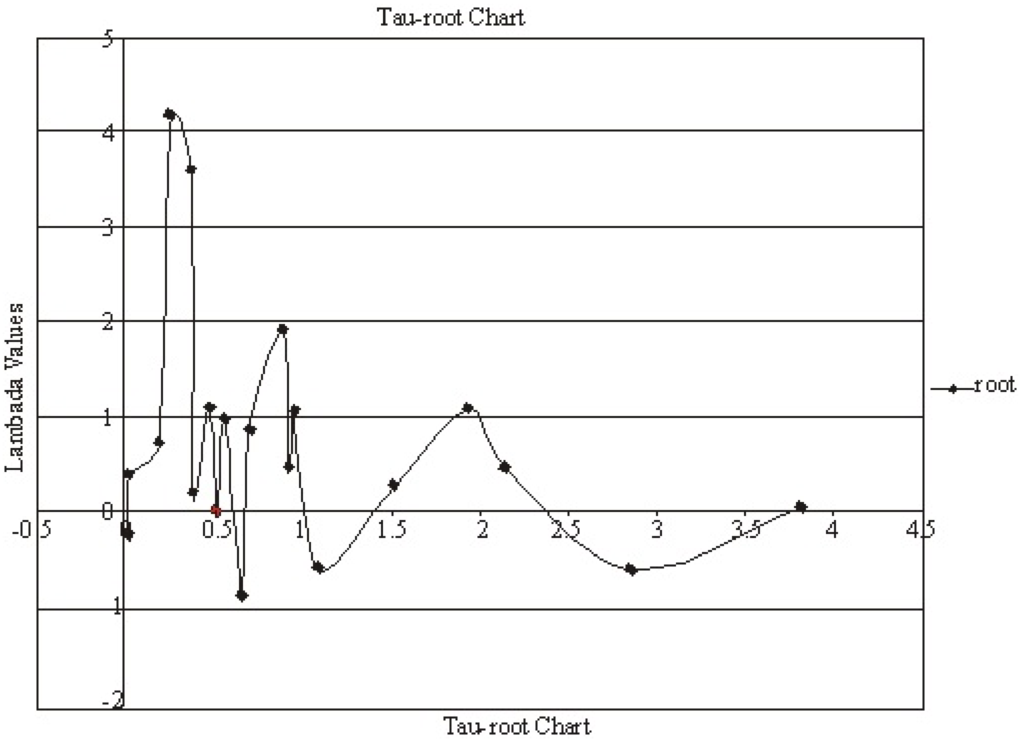

Tau-root Chart
Lambada Values
root
Tau-root Chart

Fig. 4:

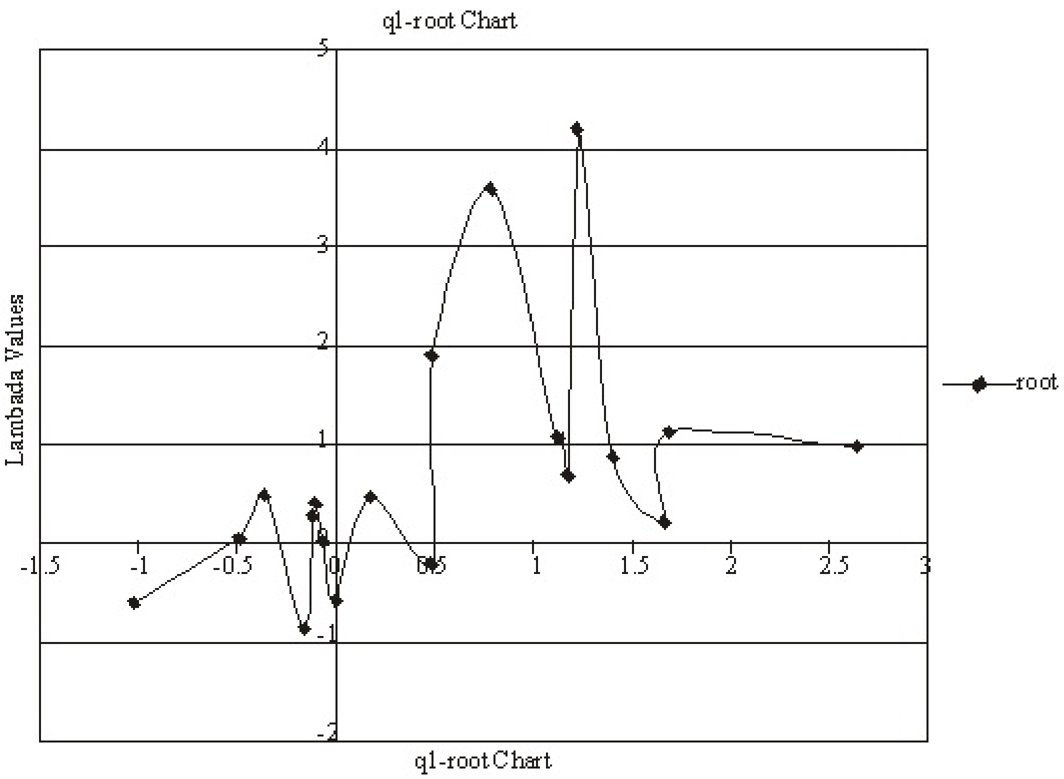
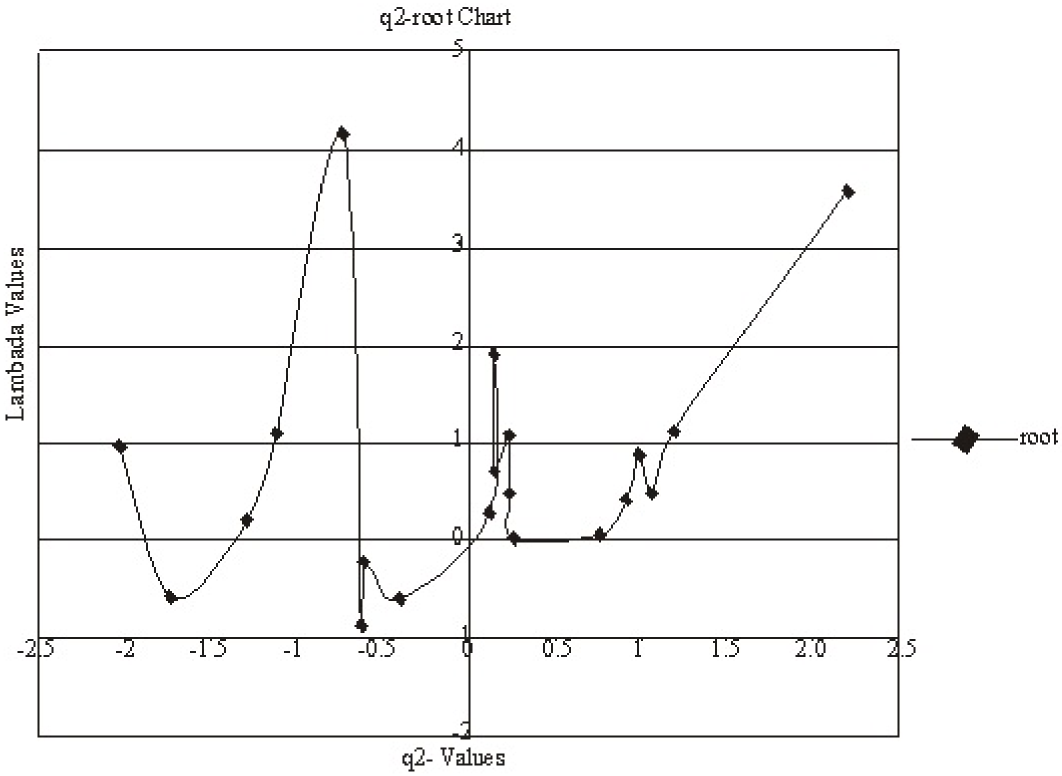

q1-root Chart
Lambada Values
root
q1-root Chart

Fig. 5:


q2-root Chart
Lambada Values
q2- Values
root
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4
3
2
1
0
-1
-2
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-2
-1.5
-1
-0.5
0
0.5
1
1.5
2.0
2.5

Fig. 6:

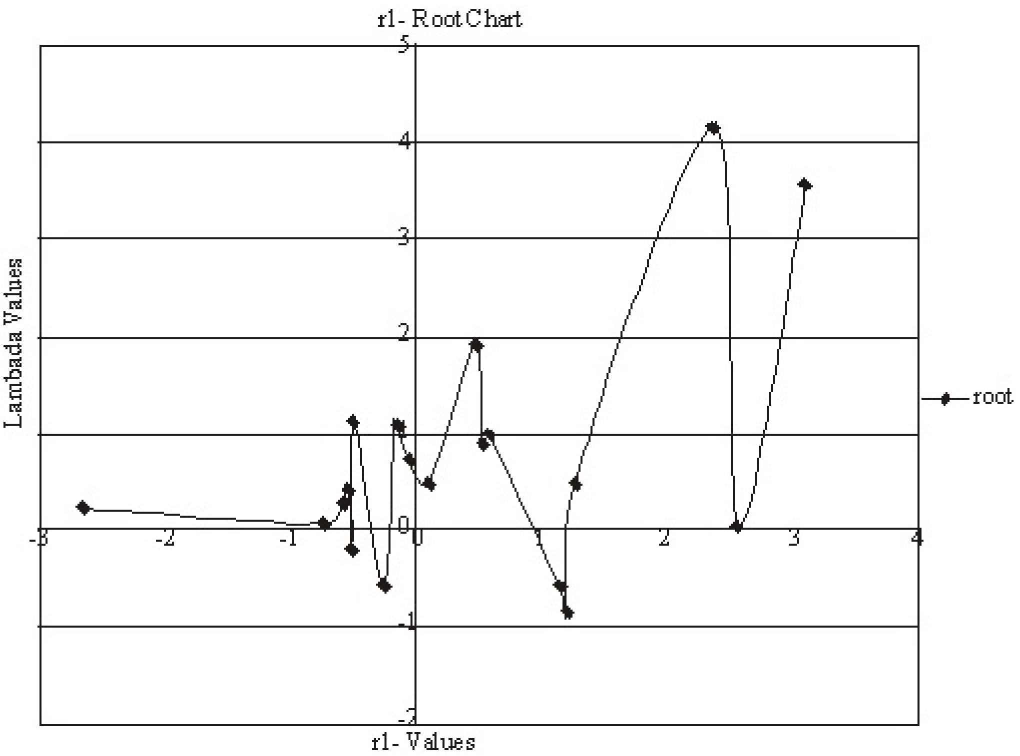
r1- Root Chart
Lambada Values
r1- Values
root
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4

Fig. 7:

Lambda - F(Lambda) Chart:

The chart between λ and F(l) in fig. 1 suggests that F(λ) values are lying very near to the zero most of the time for maximum number of the values of λ. When λ values are negative the F(λ) take positive values but as λ increases F(λ) values move to the negative domain. See fig. 1.

Further we need the comparison graphs (ii) p1 and λ (iii) p2 and λ (iv) q1 and λ (v) q2 and λ (vi) r1 and λ and (vii) τ and λ given in fig. 2 to 7 respectively to formulate the stochastic linear model realization (of the strictly determinable λ) of the form

$$\lambda = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 + \beta_6 Z_6 + \varepsilon \quad (3)$$

Remark:

Z_j's are the coefficients which are themselves suitable functions of the predictors p1, p2, q1, q2, r1 and τ. The β_j's are the parameters to be estimated by least squares. ε is the error component.

p1-values and Lambda Chart:

By observing the chart between p1 values and λ values one can easily say that as the values of the p1 goes form negative to positive, the λ values are also increasing. The correlation between p1values and λ is nonlinear and positive. See fig. 2.

p2-values and Lambda Chart:

By observing the chart between p2-values and λ values one can easily say that as the values of the p2 goes form negative to positive, the λ values are also increasing similarly to the previous Chart. The correlation between p2 values and λ is nonlinear and more positive than that of p1 values and λ (see fig. 3).

Tau and Lambda Chart:

By observing the chart between τ and λ values one can easily say that as the values of the τ increases form zero, the λ values are decreasing on an average. The correlation between t and λ is nonlinear and negative. See fig. 4.

q1-values and Lambda Chart:

By observing the chart between q1-values and λ values one can easily say that as the values of the q1 goes form negative to positive, λ are also increasing. The correlation between q1values and λ is nonlinear and positive. See fig. 5.

q2-values and Lambda Chart:

By observing the chart between q2-values and λ values one can easily say that as the values of the q2 goes form negative to positive, lvalues are also increasing. The correlation between q2values and λ is nonlinear and positive. See fig. 6.

r1-values and Lambda Chart:

By observing the chart between r1-values and λ values one can easily say that as the values of the r1 goes form negative to positive, λ values are also increasing. The correlation between r1-values and λ is nonlinear and positive. See fig. 7.

Conclusion:

We conclude that it is quite possible to create an interface between researches in computer experiments with that in differential equations to get much valuable information on the behavior of solution. Similar idea has been expressed by Fang *et. al.* (2006). The statistical approach has given us a new methodology to generate the dominant root of the second order neutral differential equation by simulating the other parameters within the prescribed interval. The dominant root decides the asymptotic stability of solution of the considered differential equation.

We are investigating the choice of the functional forms of the predictors for representing the coefficients in the model (3) using exploratory data analysis on the figures 2-7. Once an appropriate stochastic model is found, a sound platform for further work can be obtained. [Concluded]

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