

Sensitivity Analysis of Two – Level Fractional Programming Problems

¹S. Yousef and ²N. Badra

¹Department of Structural Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt.

²Department of Physics and Engineering Mathematics, Faculty of Engineering, Ain Shams University, Cairo, Egypt.

Abstract: This paper presents measurement of sensitivity for changes of violations in multiobjective linear fractional programming problems. The problem has been formulated where the objective functions are fuzzy subject to both fuzzy as well as deterministic constraints. A sensitivity analysis is carried out for the problem and investigations have been formed on the measurement of sensitivity for change in violations in the multiobjective linear fractional programming problems. The procedure is applied to a production problem of two- level programming problems where the profitability maximization problem is considered for the two – level firms.

Key words: Optimization; Production problems; Two-level linear fractional programming problems; Multicriteria optimization; Sensitivity analysis.

INTRODUCTION

In most practical applications of mathematical programming the possible values of the parameters required in the modeling of the problem are provided by either fuzzy or crisp data. Sakawa *et al* (1994). have proposed a fuzzy methodology to solve multi – level programming problems. They presented a modified method in which a satisfactory solution closer to the ideal is obtained. (Lai, 1996). has presented traditional approaches include vertex enumeration algorithm, approaches based on Khon – Tucker conditions and penalty functions. Lai and Lee (1990). have proposed a two – step fuzzy approach based on the ideal and negative ideal solutions. They have formulated the fuzzy version of general multicriteria programming problems and analyzed it. Sakawa *et al* (1999). has proposed interactive fuzzy multicriteria programming for multi – level 0-1 programming problems through genetic algorithms. Sakawa *et al* (1996). has dealt with fuzzy method for the solution of multiobjective linear continuous optimal control problems. Sinha (2003). has proposed a fuzzy programming approach to multi – level programming problems. He has suggested fuzzy mathematical programming method for the minimization of the objectives using membership functions. Sakawa *et al* (2001). have presented a case study of interactive fuzzy programming for two – level linear and linear fractional production and assignment problems. Gupta *et al* (2001). have presented sensitivity for change of violations of the aspiration levels for fuzzy multiobjective linear fractional programming problems. Sakawa *et al* (2002). have proposed an interactive approach to solve decentralized two – level transportation problem in a housing material manufacturer.

This paper presents a sensitivity analysis for two – level linear multiobjective fractional programming problems. The problem has been formulated where the objective functions are fuzzy subject to both fuzzy as well as deterministic constraints. A sensitivity analysis is carried out for the problem and investigations have been formed on the measurement of sensitivity for change in violations in the multiobjective linear fractional programming problems. The procedure is applied to a production problem of two- level programming problems where the profitability maximization problem is considered for the two – level firms. In such two – level production problem, the upper level firm tries to maximize the profitability under limited production and resource constraints. However, when a parent firm orders to its low level firm and there exists a business relationship extending over a long period of time between the two firms, it is natural that the two firms not only optimize their own objectives but also make decisions cooperatively by balancing satisfaction of one firm with that of the other. The upper level firm determines the production number of products and orders processing parts to the other lower level firm in compliance with the production planning. The two – level multiobjective linear fractional programming production problem is formulated in which the profitability is to

be maximized. Then sensitivity analysis is discussed for the problem for change in violations in the multiobjective linear fractional programming problems.

Problem Formulation:

The upper level firm determines the production number of products and orders processing parts to the other lower level firm in compliance with the production planning. The two – level multiobjective linear fractional programming production problem is formulated in which the profitability is to be maximized. The profitability of the two firms is presented by the ratios of net profit to sales or processing cost FLPP as follows:

$$(FLPP): \quad \text{Max} \quad \left(Z_{\text{Upper level}} = \frac{P^U X}{C^U X}, \quad Z_{\text{Lower level}} = \frac{P^L X - D}{C^L X} \right) \quad (1)$$

subject to

$$A X \leq R_1 \quad (2)$$

$$B X \leq R_2 \quad (3)$$

$$X^{\min} \leq X \leq X^{\max} \quad (4)$$

where,

X is a column vector of m dimensional representing the m kinds of products,

X^{\min} is a column vector of m dimensional representing the minimum requirements of products,

X^{\max} is a column vector of m dimensional representing the maximum requirements of products,

P^U is a row vector of m dimensional representing the net profit of the upper level firm,

C^U is a row vector of m dimensional representing the sales of the upper level firm,

P^L is a row vector of m dimensional representing the net profit of the lower level firm,

C^L is a row vector of m dimensional representing the processing cost of the lower level firm,

D is the labor cost of the lower level firm,

R_1 is the allowable Resource1,

A is a row vector of m dimensional representing requirements from Resource1 for each product,

R_2 is the allowable Resource2, and

B is a row vector of m dimensional representing requirements from Resource2 for each product,

Charnes and Cooper (1962). have obtained the equivalent linear programming ELPP problems for these fractional programming problems as follows:

$$(ELPP): \quad \text{Max} \quad \left(Z_{\text{Upper level}} = P^U Y, \quad Z_{\text{Lower level}} = P^L Y - D h \right) \quad (5)$$

subject to

$$A Y - R_1 h \leq 0 \quad (6)$$

$$B Y - R_2 h \leq 0 \quad (7)$$

$$C^U Y + h \leq 1 \quad (8)$$

$$C^L Y + h \leq 1 \quad (9)$$

$$A Y - R_1 h \leq 0 \quad (10)$$

$$B Y - R_2 h \leq 0 \quad (11)$$

$$Y - X^{\max} h \leq 0 \quad (12)$$

$$Y - X^{\min} h \geq 0 \quad (13)$$

Sensitivity Analysis:

If fuzziness occurs in the objective function, then the membership function of the i th fuzzy objective is:

$$\lambda_i (X) = \begin{cases} 1 & \text{if } Z_i(x) \geq Z_i^{\max} \\ \frac{Z_i(x) - Z_i^{\min}}{Z_i^{\max} - Z_i^{\min}} & \text{if } Z_i^{\max} \leq Z_i(x) \leq Z_i^{\min} \\ 0 & \text{if } Z_i(x) \leq Z_i^{\min} \end{cases} \quad (14)$$

If fuzziness occurs in the constraint, then the membership function of the j th fuzzy constraint is:

$$\lambda_j (X) = \begin{cases} 1 & \text{if } a_j x - b_j \leq 0 \\ 1 - \frac{t_j}{p_j} & \text{if } a_j x - b_j = t_j, t_j \geq 0 \\ 0 & \text{if } a_j x - b_j \geq p_j, t_j \leq p_j \end{cases} \quad (15)$$

where,

t_j is the allowed value of violation in the j th constraint, and
 p_j is the maximum value of the violation in the j th constraint.

If fuzziness occurs in the problem (ELPP) in the objective functions of both upper and lower level firm, then the problem with sensitivity analysis (SELPP) can be written as follows:

$$\text{(SELPP):} \quad \text{Max } (\lambda = \sum_{i=1}^4 \frac{\lambda_i}{4}) \quad (16)$$

subject to:

$$P^U Y - (Z_U^{\max} - Z_U^{\min}) \lambda_1 \geq Z_U^{\min} \quad (17)$$

$$P^L Y - D h - (Z_L^{\max} - Z_L^{\min}) \lambda_2 \geq Z_L^{\min} \quad (18)$$

$$\lambda_1 \geq \lambda^* \text{ (allowable-upper)} \quad (19)$$

$$\lambda_1 \geq \lambda_2 \quad (20)$$

$$A Y - R_1 h \leq 0 \quad (21)$$

$$B Y - R_2 h \leq 0 \quad (22)$$

$$C^U Y + h - t_1 = 1 \quad (23)$$

$$C^L Y + h - t_2 = 1 \quad (24)$$

$$Y - X^{\max} h \leq 0 \quad (25)$$

$$Y - X^{\min} h \geq 0 \quad (26)$$

$$P_1 \lambda_3 + t_1 \leq P_1, t_1 \leq P_1 \quad (27)$$

$$P_2 \lambda_4 + t_2 \leq P_2, t_2 \leq P_2 \quad (28)$$

$$Y, \lambda_i, t_1, t_2 \geq 0, \lambda_i \leq 1, i=1,2,3,4 \quad (29)$$

Note that condition (20) is defined to ensure that degree of satisfaction for the upper level firm is allows greater than that of the lower level firm. The upper level firm determines the minimum level of satisfaction for the firm to be λ^* (allowable-upper). The optimal value is denoted by λ_{\max} .

Changing P_1 to $P_1 + \Delta P_1$ a new equivalent solution of (SELPP) is obtained with the solution given by λ_{\max}^{Δ} . The change of the optimal value can be calculated as $(\lambda_{\max}^{\Delta} - \lambda_{\max})$. Also, when changing P_2 to $P_2 + \Delta P_2$ another new equivalent solution of (SELPP) is obtained with the solution given by λ_{\max}^{Δ} . The change of the optimal value can be calculated as $(\lambda_{\max}^{\Delta} - \lambda_{\max})$.

Illustrative Example:

Consider the following two – level linear fractional programming production problem (FLPP1) :

$$\text{(FLPP1):} \quad \text{Max } (Z_{\text{Upper level}} = \frac{P^U X}{C^U X}, Z_{\text{Lower level}} = \frac{P^L X - D}{C^L X})$$

subject to

$$\begin{aligned} A X &\leq R_1 \\ B X &\leq R_2 \\ X^{\min} &\leq X \leq X^{\max} \end{aligned}$$

where,

$$\begin{aligned} P^U &= [70 \quad 100 \quad 40] , & X &= [x_1 \quad x_2 \quad x_3]^t , & C^U &= [80 \quad 150 \quad 50] \\ P^L &= [10 \quad 50 \quad 10] , & D & \text{(fixed labor cost) = 2500} , & C^L &= [20 \quad 90 \quad 25] \\ A &= [40 \quad 50 \quad 20] , & R_1 & \text{(allowable resource 1) = 18000} \\ B &= [70 \quad 20 \quad 10] , & R_2 & \text{(allowable resource 2) = 12000} \\ X^{\min} &= [50 \quad 50 \quad 5]^t , & X^{\max} &= [200 \quad 400 \quad 20]^t , & X &\geq 0 \end{aligned} \quad (30)$$

Results of single objective optimization for the upper and lower level firms are given in Table 1.

Table 1: Results of single objective optimization for the upper and lower level firms.

Z_U^{max}	Z_U^{min}	$Z_U^{max} - Z_U^{min}$	Z_L^{max}	Z_L^{min}	$Z_L^{max} - Z_L^{min}$
0.797	0.683	0.114	0.469	0.098	0.371

The equivalent linear programming (ELPP1) problem for this fractional programming problem is as follows:

$$(ELPP1): \quad \text{Max} \quad (Z_{Upper\ level} = P^U Y , Z_{Lower\ level} = P^L Y - 2500 h) \quad (31)$$

subject to

$$A Y - 18000 h \leq 0 \quad (32)$$

$$B Y - 12000 h \leq 0 \quad (33)$$

$$C^U Y + h \equiv 1 \quad (34)$$

$$C^U Y + h \equiv 1 \quad (35)$$

$$Y - X^{max} h \leq 0 \quad (36)$$

$$Y - X^{min} h \geq 0 \quad (37)$$

$$Y = [y_1 \quad y_2 \quad y_3]^t , \quad Y , h \geq 0 \quad (38)$$

The equivalent two – level linear programming problem with sensitivity analysis (SELPP1) can be written as follows:

$$(SELPP1): \quad \text{Max} \quad (\lambda = \sum_{i=1}^4 \frac{\lambda_i}{4}) \quad (39)$$

subject to:

$$P^U Y - 0.114 \lambda_1 \geq 0.683 \quad (40)$$

$$P^L Y - 2500 h - 0.371 \lambda_2 \geq 0.098 \quad (41)$$

$$\lambda_1 \geq 0.6 \quad (42)$$

$$\lambda_1 \geq \lambda_2 \quad (43)$$

$$A Y - 18000 h \leq 0 \quad (44)$$

$$B Y - 12000 h \leq 0 \quad (45)$$

$$C^U Y + h - t_1 = 1 \quad (46)$$

$$C^L Y + h - t_2 = 1 \quad (47)$$

$$Y - X^{max} h \leq 0 \quad (48)$$

$$Y - X^{min} h \geq 0 \quad (49)$$

$$P_1 \lambda_3 + t_1 \leq P_1 , \quad t_1 \leq P_1 \quad (50)$$

$$P_2 \lambda_4 + t_2 \leq P_2 , \quad t_2 \leq P_2 \quad (51)$$

$$Y , \lambda_i , t_1 , t_2 \geq 0 , \quad \lambda_i \leq 1 , \quad i=1,2,3,4 \quad (52)$$

Changing P_1 to $P_1 + \Delta P_1$ a new equivalent of (SELPP1) is obtained and changing P_2 to $P_2 + \Delta P_2$ another new equivalent of (SELPP1) is obtained.

Tables 2-3 show the changes in the optimal value λ^{max} of problem (SELPP1).

Table 2: Changes in P_1 of the optimal value λ^{max} of problem (SELPP1).

P_1	λ^{max}	y_1	y_2	y_3	h	λ_1	λ_2	λ_3	λ_4	t_1	t_2
3	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
5	0.963	0.002	0.011	0	0	1	1	0.851	1	0.746	0
15	0.988	0.002	0.011	0	0	1	1	0.950	1	0.746	0
50	0.996	0.002	0.011	0	0	1	1	0.985	1	0.746	0
100	0.998	0.002	0.011	0	0	1	1	0.993	1	0.746	0
1000	0.999	0.002	0.011	0	0	1	1	0.999	1	0.746	0
50000	0.999	0.002	0.011	0	0	1	1	0.999	1	0.746	0

Table 3: Changes in P_2 of the optimal value λ^{max} of problem (SELPP1).

P_2	λ^{max}	y_1	y_2	y_3	h	λ_1	λ_2	λ_3	λ_4	t_1	t_2
3	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
5	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
15	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
50	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
100	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
1000	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0
50000	0.938	0.002	0.011	0	0	1	1	0.751	1	0.746	0

It has been noted that the current optimal solution becomes unchanged after P_1 achieve certain value (Table 3) or the current optimal solution may remain unchanged (Table 4).

Figures 1 and 2 show the effect of changes in P_1 and P_2 in the current optimal solution, respectively.

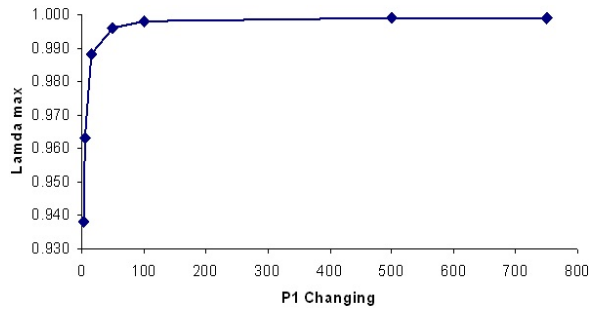


Fig. 1: The effect of changes in P_1 in the current optimal solution λ_{max} .

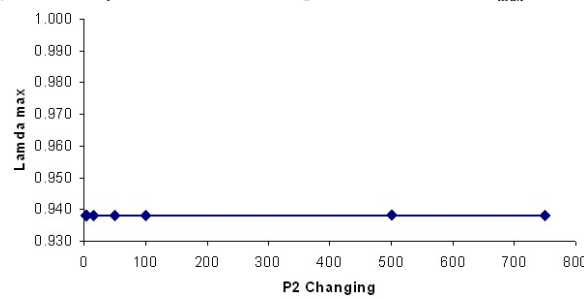


Fig. 2: The effect of changes in P_2 in the current optimal solution λ_{max} .

Conclusion:

This paper presents a sensitivity analysis for two – level linear multiobjective fractional programming problems. A sensitivity analysis is carried out for the problem and investigations have been formed on the measurement of sensitivity for change in violations in the multiobjective linear fractional programming problems. The procedure is applied to a production problem of two- level programming problems where the profitability maximization problem is considered for the two – level firms. In such two – level production problem, the upper level firm tries to maximize the profitability under limited production and resource constraints. The two – level multiobjective linear fractional programming production problem is formulated in which the profitability is to be maximized. Then sensitivity analysis is discussed for the problem for change in violations in the multiobjective linear fractional programming problems. The equivalent linear objective functions of both upper and lower level firm is formulated, and then the equivalent problem with sensitivity analysis is discussed. The effect of changing P_1 to $P_1 + \Delta P_1$ and P_2 to $P_2 + \Delta P_2$ on the current optimal solution is presented. It has been found that the current optimal solution may become unchanged after the violation achieving certain value or the current optimal solution may remain unchanged.

REFERENCES

Charnes, A. and W. Cooper, 1962, " Programming with Linear Fractional Functionals ", Naval Res. Logist. Quart., 9: 181-186.

Gupta, P. and D. Bhatia, 2001, " Sensitivity Analysis in Fuzzy Multiobjective Linear Fractional Programming Problem ", Fuzzy Sets and Systems, 122; 229-236.

Lai, Y. 1996 " Hierarchical Optimization: A Satisfactory Solution ", Fuzzy sets and Systems, 77: 321-335.

Lai, R. and E. Lee, 1990, " Fuzzy Approaches to Multicriteria de Novo Programs ", Journal Math., Anal. Appl., 153: 97-111.

Sakawa, M. and H. Yano, 1994, " A Fuzzy Dual Decomposition Method for Large – Scale Multiobjective Nonlinear Programming Problems ", Fuzzy sets and Systems, 67: 19-27.

Sakawa, M., I. Nishizaki and M. Hitaka, 1999, " Interactive Fuzzy Programming for Multi – level 0 - 1 Programming Problems through Genetic Algorithms ", European Journal of Operational Research, 114: 580-588.

Sakawa, M., M. Inuiguchi, K. Kato and T. Ikeda, 1996, " A Fuzzy Satisficing Method for multiobjective Linear Optimal Control Problems ", *Fuzzy Sets and Systems*, 78: 223-229.

Sinha, S., 2003, " Fuzzy Programming Approach to Multi – level Programming Problems ", *Fuzzy Sets and Systems*, 136: 189-202.7.

Sakawa, M., I. Nishizaki and Y. Uemura, 2001, " Interactive Fuzzy Programming for Two – level Linear and Linear Fractional Production and Assignment Problems: A Case Study ", *European Journal of Operational Research*, 135: 142-157.

Sakawa, M., I. Nishizaki and Y. Uemura, 2002, " A Decentralized Two – level Transportation Problem in a Housing Material Manufacturer: Interactive Fuzzy Programming Approach ", *European Journal of Operational Research*, 141: 167-185.