

Different Approaches to the Solution of Damped Forced Oscillator Problem by Decomposition Method

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Abstract: In this work, we first use the standard Adomian decomposition method (ADM) to solve the damped forced oscillator problem as a second order ordinary differential equation. Then the equation is converted to a system of two first order ordinary differential equations and solved by ADM and Runge-Kutta method (RKM). The numerical results show that the ADM is efficient and easy to use.

Key words: Adomian decomposition method; Damped forced oscillator; Runge- Kutta method

INTRODUCTION

Adomian decomposition method has been extensively used to solve linear and nonlinear problems arising in many interesting physical and engineering applications (Adomian, 1989; Adomian, 1986; Adomian, 1984; Deeba and Khuri, 1996; Deeba and Khuri, 1996). In this method the unknown quantity is replaced with a series. It has been shown that (Cherruault *et al.*, 1992) the series converges fast and, with only a few terms, approximates the exact solution with a small error normally less than one percent.

Since every ordinary differential equation (ODE) of the order n , can be converted to a system consisting of n ordinary differential equations of order one and be solved analytically (Simmons, 1972) or numerically (Richard, 2001; Biazar *et al.*, 2004), in this work, we convert the equation of damped forced oscillator to a system of two first order ordinary differential equations and solve it using ADM and RKM.

The methods:

In this section, we first explain the ADM for damped forced oscillator problem as a second order ODE, then convert the equation to a system of first order ODEs and solve it using ADM. Finally the well known RKM for a system of differential equations is briefly introduced and applied to solve the system of equations obtained from damped forced oscillator problem.

The ADM to DFO:

The damped forced oscillator problem is described by a second order ordinary differential equation with the common form

$$mx'' + bx' + kx = F(t), \quad x(0) = x_0, \quad x'(0) = x'_0, \quad (1)$$

in which m , b and k are mass, the viscosity coefficient of the fluid and the stiffness of the spring, respectively, and $F(t)$ is a time-dependent external force. Denoting $\frac{d^2}{dt^2}$ by L , we have L^{-1} as a two-fold integration from 0 to t . Therefore, equation (1) can be written as:

$$mLx = -bx' - kx + F(t) \quad (2)$$

Operating with L^{-1} ,

$$L^{-1}Lx = -\frac{b}{m} L^{-1}x' - \frac{k}{m} L^{-1}x + \frac{1}{m} L^{-1}F(t), \tag{3}$$

$$x = x(0) + tx'(0) - \frac{b}{m} L^{-1} \frac{d}{dt} x - \frac{k}{m} L^{-1}x + \frac{1}{m} L^{-1}F(t). \tag{4}$$

To use ADM, let $\sum_{r=0}^{\infty} x_r$. Hence, from (4) we have

$$x_0 = x(0) + tx'(0) + \frac{1}{m} L^{-1}F(t), \tag{5}$$

$$x_r + 1 = -\frac{b}{m} L^{-1} \frac{d}{dt} x_r - \frac{k}{m} L^{-1}x_r, \quad r = 0, 1, 2, \dots \tag{6}$$

In practice, not all the terms of the series $x(t) = \sum_{r=0}^{\infty} x_r(t)$, can be determined and so we approximate the solution by the following truncated series

$$\varphi_k(t) = \sum_{r=0}^{k-1} x_r(t), \quad \lim_{k \rightarrow \infty} \varphi_k(t). \tag{7}$$

For the convergence of the Adomian decomposition scheme, we refer the reader to (Adomian, 1989; Cherruault, 1989; Cherruault, 1992).

Applying ADM to a system of ODES:

Most applied problems are described by second-order and higher-order differential equations. A differential equation of order n , can be written as

$$x^{(n)} = f_i(x, y, x'', \dots, x^{(n-1)}); \quad n \geq 2 \tag{8}$$

with $x^{(n)}(0) = x_{0n}$ as initial conditions. Using $x^{(j)} = y_{j+1}$ this equation converts to a system of ordinary first-order differential equations as follows

$$y_i' = f_i(x, y_1, \dots, y_n^{(n-1)}); \quad y_i(x_0) = y_{0i}, \quad i = 1, 2, \dots, n, \tag{9}$$

where each equation represents the first derivative of a function as a map depending on the independent variable x , and n unknown functions f_1, \dots, f_n . As we plan to solve a linear system of differential equations that govern DFO, system (9) is linear. Defining the operator L as the first order derivative with respect to t , the i -th equation of system (9) can be represented as the common form

$$Ly_i = f_i(x, y_1, \dots, y_n) \tag{10}$$

Applying the inverse of L , $L^{-1}(\cdot) = \int_0^t (\cdot) dt$, equation (10) can be written as

$$y_i = y(0) + \int_0^t f_i(x, y_1, \dots, y_n) dt, \tag{11}$$

which is called the canonical form in Adomain scheme. In order to apply the Adomaindecomposition method, we let

$$y_i = \sum_{j=0}^{\infty} y_{ij}, \tag{12}$$

$$f_i(x, y_1, \dots, y_n) = \sum_{k=1}^n \sum_{j=0}^{\infty} a_k y_{ij}^j, \tag{13}$$

where $a_k, k = 0, 1, \dots, n$ are scalars. Substituting (12), (13) into (11) yields

$$\sum_{j=0}^{\infty} y_{ij}^j = y_i(0) + \int_0^t \sum_{k=1}^n \sum_{j=0}^{\infty} a_k y_{ij}^j dt, \tag{14}$$

from which we define

$$\begin{cases} y_{i0} = y_i(0) \\ y_{ij} + 1 = \int_0^t \sum_{k=1}^n a_k y_{ij}^j dt, \quad j = 0, 1, \dots \end{cases} \tag{15}$$

In practice, not all terms of the series $y_i(t) = \sum_{j=0}^{\infty} y_{ij}^j(t)$, can be determined. So we consider an approximate solution, calculating the following truncated series

$$\varphi_{ik}(t) = \sum_{j=0}^{k-1} y_{ij}^j(t), \quad \lim_{k \rightarrow \infty} \varphi_{ik}(t) = y_i. \tag{16}$$

Our procedure leads to a system of second kind Volterra integral equations, so referring to (Babolian and Biazar, 2001), the convergence of the method is proved.

Runge-Kutta Method (RKM):

In the Runge-Kutta method, an approximation to dependent variable in $x_i + \Delta t$, is obtained from x_i , in such a way that the power series expansion of the approximation coincides, up to terms of a certain order $(\Delta t)^N$ in the time interval Δt , with the actual Taylor series expansion of $t + \Delta t$ in powers of Δt . The method is based on the assumption that the higher derivatives exist at required points. The Runge-Kutta method is self starting and has the advantage that no initial values are needed beyond the prescribed values. A brief discussion of its basis is presented here.

In the Runge-Kutta method the second order differential equation is first reduced to two first order equations. Consider the differential equation of damped forced oscillator which can be written as

$$x'' = \frac{1}{m} [F(t) - bx' - kx] = f(x, x', t) \tag{17}$$

Choosing $x_1' = x$ and $x_2 = x'$, the equation is reduced to the following two first order equations. By defining $\bar{x}(t) = (x_1(t), x_2(t))^T$ and $\bar{H}(t) = (x_2(t), f(x_1, x_2, t))^T$, the following recurrence formula is obtained

to the values of $x(t)$ at mesh or grid points it according to the fourth order Runge-Kutta method. We omit the details of the derivation of the method.

$$\bar{x}_i + 1 = \bar{x}_i + \frac{1}{h} [\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4] \tag{18}$$

where $\bar{K}_1 = h\bar{H}(\bar{x}_i, t_i)$, $\bar{K}_2 = h\bar{H}(\bar{x}_i + \frac{1}{2}\bar{K}_1, t_i + \frac{1}{2}h)$, $\bar{K}_3 = h\bar{H}(\bar{x}_i + \frac{1}{2}\bar{K}_2, t_i + \frac{1}{2}h)$, $\bar{K}_4 = h\bar{H}(\bar{x}_i + \frac{1}{2}\bar{K}_3, t_i + 1)$

The main drawback of the method is that each forward step requires several computations of the functions, thus increasing the computational cost.

Application:

Consider the damped forced oscillator problem with the equation (Richard *et al.*, 2001).

$$x'' - 2x' + 2x = e^{2t} \sin t, \quad x(0) = -0.4, \quad x'(0) = -0.6, \quad 0 \leq t \leq 1, \tag{19}$$

Using classical methods (Simmons, 1972), the exact solution of equation (19) is

$x(t) = 0.2e^{2t}(\sin t - 2 \cos t)$. In this section, this equation is solved using ADM and then, choosing

$y_1(t) = x(t)$ and $y_2(t) = x'(t)$, converts to a system of first order differential equations as

$$\begin{cases} y_1'(t) = y_2(t), & y_1(0) = -0.4, \\ y_2'(t) = e^{2t} \sin(t) - 2y_1(t) + 2y_2(t), & y_2(0) = -0.6, \end{cases} \tag{20}$$

and solved using ADM. Finally the RKM will be applied to (20).

i)Applying ADM to (19)

Denoting $\frac{d^2}{dt^2}$ by L , we have L^{-1} as a two-fold integration from 0 to t . Therefore, equation (19) can be written as

$$Lx = e^{2t} \sin(t) + 2x' - 2x \tag{21}$$

Operating with L^{-1} ,

$$x = L^{-1}e^{2t} \sin(t) + 2L^{-1}x' - 2L^{-1}x, \tag{22}$$

To use ADM, let $x = \sum_{r=0}^{\infty} x_r$. Hence, from (22) we have

$$x_0 = x(0) + tx'(0) + \frac{1}{m} L^{-1}e^{2t} \sin(t) \tag{23}$$

$$x_{r+1} = 2L^{-1} \frac{d}{dt} x_r - 2L^{-1}x_r, \quad r = 0, 1, \dots \tag{24}$$

To approximate the solution as a polynomial for the special case introduced in section 2, replacing $e^{2t} \sin(t)$ with the first four terms of its Maclaurin series and using the initial conditions of the problem in equations (23), we obtain

$$x_0 = -0.4 - 0.6t + \int_0^t \int_0^t (t + 2t^2 + \frac{11t^3}{6} + t^4) dt dt \tag{25}$$

$$= -0.4 - 0.6t + \frac{t^3}{6} + \frac{t^4}{6} + \frac{11t^5}{120} + \frac{t^6}{30} \tag{26}$$

$$x_{r+1} = 2 \int_0^t \int_0^t \frac{d}{dt} (x_r) dt dt - 2 \int_0^t \int_0^t x_r dt dt, \quad r = 0, 1, \dots \tag{27}$$

In a similar manner, the components x_r are calculated for $r = 0, 1, \dots$, but for brevity they will not be listed here. Considering (7), the approximate solution including eight terms is

$$x_{ADM}(t) \approx \varphi_8 = -0.4 - 0.6t - 0.2t^2 + 0.233333t + 0.316667t^4 + 0.195t^5 \dots - 1.26269 \times 10^{-15} t^{20}$$

Using this polynomial, the approximations of x values for $t = k/10, k = 0, 1, \dots, 10$ can be obtained. The absolute errors of this method are shown in Table 1, labeled by ADM-Error.

ii) Applying ADM to (20)

In order to apply the ADM to (20), we let $y_i = \sum_{j=0}^{\infty} y_{ij}$ to obtain

$$\begin{cases} y_{10} = -0.4, \\ y_{1,j+1} = \int_0^t y_{2,j} dt, \quad j = 0, 1, 2, \dots \end{cases} \quad (29)$$

and

$$\begin{cases} y_{2,0} = -0.6 + \int_0^t (t + 2t^2 + \frac{11t^3}{6} + t^4) dt, \\ y_{2,j+1} = -2 \int_0^t y_{1,j} dt + 2 \int_0^t y_{2,j} dt dt, \quad j = 0, 1, 2, \dots \end{cases} \quad (30)$$

in which we use four terms of $e^{2t} \sin t$ in the second equation of (20). Following the procedure introduced in section 2.2 and calculating eight terms of (16) for $i = 1$, the approximate solution of the problem is obtained as

$$y_1(t) = x(t) \approx \varphi_8 = -0.4 - 0.6t - 0.2t^2 + 0.233333t^3 + \dots - 4.00834 \times 10^{-7} t^{12} \quad (31)$$

The absolute errors of this approach are shown in Table 1 as SYS-ADM-Error.

iii) Applying RKM to (20)

To apply the RKM, we use relation (18) to obtain the numerical solutions of equation (20). The errors of this method for different points in the interval $[0, 1]$ are shown in Table 1. To study this method, in more detail, we refer the reader to (20). The related errors are shown in Table 1, labeled by RKM-Error.

RESULTS AND DISCUSSIONS

To have a comparison of the methods employed, we report the errors of the three methods applied for different values of t in Table 1.

Table 1: The exact solution and absolute errors of the methods applied.

Time	Exact	ADM-Error	SYS-ADM-Error	RKM-Error
0	-0.4000			
0.1	-0.461733	8.44×10^{-10}	8.43×10^{-10}	3.70×10^{-7}
0.2	-0.525559	1.12×10^{-7}	1.11×10^{-7}	8.30×10^{-7}
0.3	-0.588602	1.98×10^{-6}	1.98×10^{-6}	1.39×10^{-6}
0.4	-0.646626	1.54×10^{-5}	1.53×10^{-5}	2.03×10^{-6}
0.5	-0.693640	7.62×10^{-5}	7.57×10^{-5}	2.71×10^{-6}
0.6	-0.721431	2.82×10^{-3}	2.80×10^{-3}	3.41×10^{-6}
0.7	-0.719008	8.59×10^{-4}	8.51×10^{-4}	4.05×10^{-6}
0.8	-0.671966	2.25×10^{-3}	2.23×10^{-3}	4.56×10^{-6}
0.9	-0.561752	5.31×10^{-3}	5.25×10^{-3}	4.76×10^{-6}
1	-0.364839	1.14×10^{-2}	1.13×10^{-2}	4.50×10^{-6}

As can be seen, the errors of ADM and sys-ADM are nearly the same, but it is noticeable that the response polynomial for the ADM is of the order 20 while for the sys-ADM is of the order 12.

The error of these methods in $t = 0.1$ is about 10^{-10} and at the end of the interval it is about 10^{-2} . The accuracy of the result in the beginning of the interval is related to the accuracy of the Taylor series at these points. The error of RKM is about 10^{-6} to 10^{-7} , which has a more suitable distribution on the interval. While Adomian's decomposition scheme is simpler, the RKM gives more accurate results for DFO problem.

Conclusion:

In this work, we used three numerical methods to approximate the solutions of differential equations that govern the oscillative systems, and compared the results with the exact solution. The numerical results show that the RKM gives a uniform error in the interval, while the ADM gives more accurate results than the RKM in the beginning of the interval and its error increases as the time goes by. This is due to using the Taylor series of $F(t)$. In Adomian decomposition scheme, converting the differential equation to a system of differential equations of lower order tends to yield a response polynomial of lower order.

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