

Improving the Accuracy of Solutions of Linear and Nonlinear ODEs in Decomposition Method

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Abstract: In the solution of differential equations using Adomian's decomposition method, if there is any nonlinear function of the independent variable of the equation, its Taylor series is commonly used in order to obtain the solution of the problem as a polynomial. In this paper, we use the interpolation of these functions in the required interval in spite of their Taylor series. The numerical results show that this method in identical conditions gives more suitable solutions for linear and nonlinear differential equations than standard Adomian decomposition method.

Key words: Adomian decomposition method; Differential equation; Interpolation; Damped forced oscillator; Duffing oscillator

INTRODUCTION

The subject of the Adomian decomposition method (ADM) has been rapidly growing in recent years. The concept of this method was first introduced by G. Adomian at the beginning of the 1980's (Adomian, G., 1989; Adomian, G., 1994). ADM consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as the initial approximation, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and solving the obtained recursive equation.

Several authors have proposed a variety of modifications to ADM. Wazwaz proposed a powerful modification of ADM that will accelerate the rapid convergence of the series solution (Wazwaz, A.M., 1999). E. Babolian et al. used restarted method to solve the equation $f(x) = 0$ (Babolian, E., S.H. Javadi, 2003) and integral equations (Babolian, E., S.H. Javadi, *et al*, 2003). Hossein Jafari et al. used a correction of decomposition method for ordinary and nonlinear systems of equations and showed that the correction accelerates the convergence (Hossein Jafari, Varsha Daftardar-Gejji, 2006).

If we need to obtain approximate solutions as a polynomial, we must replace the nonlinear function of the independent variable, if any, with its Taylor series at the initial point. Obviously, the Taylor series is not the best approximation when we move away from the initial point. This approximation imposes an increasing error on the solutions as the independent variable tends to the final point. In fact, the Taylor series is an interpolated polynomial that is accurate just at the initial point. In this work, we propose a modification to the method that causes a uniform error all over the interval.

This paper is organized as follows. In section 2, the standard ADM for differential equations is explained in detail. The proposed modification is introduced in section 3. The effectiveness of the modified technique in comparison with the standard approach is shown through two examples in section 4. The results are presented and discussed in section 5. In section 6, we briefly discuss the conclusions.

2 Description of ADM:

Consider the differential equation with the common form

$$\mathcal{G}x(t) = F(t) \tag{1}$$

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where G is a general nonlinear ordinary differential operator that can involve both linear and nonlinear parts, and $F(t)$ is a given function called the excitation term. The linear term in $Gx(t)$ can be decomposed into $Lx(t)+Rx(t)$, where L is an easily invertible operator, which is taken generally as the highest-order derivative for avoiding the difficult integrations when the complicated Green's function is involved, R is the remainder of the linear operator and N is the nonlinear part of $G(t)$. Thus, equation (1) can be written as

$$Lx(t) + Rx(t) + Nx(t) = F(t) \tag{2}$$

Solving for $Lx(t)$,

$$Lx(t) = F(t) - Rx(t) - Nx(t) \tag{3}$$

As L is invertible, applying its inverse, L^{-1} , yields

$$L^{-1}Lx(t) = L^{-1}F(t) - L^{-1}Rx(t) - L^{-1}Nx(t) \tag{4}$$

or

$$x(t) = \Phi = L^{-1}F(t) - L^{-1}Rx(t) - L^{-1}Nx(t) \tag{5}$$

in which Φ is the integration constant and satisfies $L\Phi = 0$. If equation (1) corresponds to an initial value problem, the operator L^{-1} may be regarded as definite integrations from 0 to t . If L is a second order operator, then L^{-1} is a two-fold integration, and $\Phi = x(0) + x'(0)t$. By ADM (Adomian, G., 1989; Adomian, G., 1994; Wazwaz, A.M., 1999), the solution x is represented as the infinite sum of the series

$$x(t) = \sum_{n=0}^{\infty} x_n(t) \tag{6}$$

In the Adomian schema, the nonlinear part of the equation is replaced by the summation of Adomian's polynomials, whose i -th term is given by

$$A_i = \frac{1}{j!} \left[\frac{d^j}{d\lambda^j} N\left(\sum_{j=0}^{\infty} \lambda^j y_i\right) \right]_{\lambda=0} \tag{7}$$

Setting (6) in (5) and considering the nonlinear term as $Nx(t) = \sum_{n=0}^{\infty} A_n(t)$, we have

$$\sum_{n=0}^{\infty} x_n(t) = \Phi + L^{-1}F(t) - L^{-1}R\sum_{n=0}^{\infty} x_n(t) - L^{-1}\sum_{n=0}^{\infty} A_n(t) \tag{8}$$

Each term of series (8) is given by the recurrence relation

$$x_0(t) = \Phi + L^{-1}F(t) \tag{9}$$

$$x_n(t) = -L^{-1}Rx_{n-1}(t) - L^{-1}A_{n-1}(t) \tag{10}$$

In practice, not all terms of series (6) can be determined, and hence the solution will be approximated by the truncated series

$$\phi_k(t) = \sum_{n=0}^{k-1} x_n(t) \tag{11}$$

Using the Taylor series of $F(t)$ in (9), yields the approximate solution (11) as a polynomial. Although the series solutions converge rapidly only in a small region, in the wide region, they may have very slow convergence rates, and then their truncations yield inaccurate results.

Description of the Modified Technique:

Consider the differential equation (1), whose solution in $[a, b]$ is desired. Using the Taylor series of the function $F(t)$ in ADM, we obtain a polynomial as the solution of the equation. Suppose $g_n(t)$ denotes the n -th order approximation of the Taylor series of $F(t)$ around the lower end of the interval, a . The Taylor series behaviour is such that $g_n(a)$ is a good approximation of $F(a)$, but as t goes toward the upper end of the interval, b , the error $|F(t) - g_n(t)|$ increases, such that around the end of the interval, the approximation would not be acceptable and this causes an exceeding error that is considerable in the end of the interval. In this way the ADM will encounter an exceeding nonuniform error over the interval $[a, b]$.

To resolve this problem, we use the interpolated polynomial of the excitation term. Toward this end, consider the function $F(t)$ defined on the interval $[a, b]$. Splitting the interval into n equal sections of the

width $h = \frac{b-a}{n}$, we have $n + 1$ support points (t_i, F_i) , $i = 0, 1, \dots, n$, where $t_i = a + ih$ and $F_i = F(t_i)$,

then there is an n -th order unique polynomial $P(t)$ that approximates the function $F(t)$ as

$$F(t) \approx P(t) = a_0 + a_1t + \dots + a_nt^n \tag{12}$$

The approximated polynomial, $P(t)$, can be obtained by any method of interpolation (Richard, L., J. Burden, 2001). Obviously, the interpolated polynomial $P(t)$ is more suitable than the Taylor series of the function $F(t)$ in the interval because the Taylor series coincides with $F(t)$ just at one point while the interpolated polynomial $P(t)$ coincides at $n + 1$ points.

Numerical Examples:

In this section, we apply the standard ADM and the proposed method to solve two physical examples. The first is a linear second order ODE and the second is a nonlinear differential equation.

Example 1:

As the first example, the damped forced oscillator problem is solved, which is a famous physical problem that appears in a variety of physics and engineering fields (Goldstein, H., 1980). The common form of this problem is described by the differential equation

$$mx'' = -bx' - kx + F(t) \tag{13}$$

that describes a point mass connected to a spring and moving in a viscous fluid under the external excitation force $F(t)$. In this equation, m is the mass, k is the stiffness of the spring and b is the viscous damping coefficient (Thomsen, J.J., 1997). A special choice is $F(t) = e^{2t} \sin t$, $m = 1\text{kg}$, $k = 2\text{N/m}$, $b = -2\text{Ns/m}$, $x(0) = -0.4$, and $x'(0) = -0.6$ as initial conditions (Richard, L., J. Burden, 2001). Using classical methods (Simmons, G.F., 1972), the exact solution of equation (13) with the introduced coefficients will be as follows

$$x(t) = 0.2e^{2t}(\sin t - 2\cos t) \tag{14}$$

of which the numerical results for $0 \leq t \leq 1$ are presented in Table 1 as "Exact".

Standard ADM:

To solve equation (13) by ADM, we use four terms of the Maclaurin series for $F(t) = e^{2t} \sin t$ in (9) and use (10). Note that our problem is a linear ODE and so the term $L^{-1}A_{n-1}(t)$ in (10) is not included in the solution of this example. The procedure of ADM yields

$$x_0 = x(0) + tx'(0) + \int_0^t \int_0^s (t + \frac{t^2}{2} + \frac{1}{6}t^3 + t^4) dt ds \tag{15}$$

$$= -0.4 - 0.6t + \frac{t^3}{6} + \frac{t^4}{6} + \frac{11t^5}{120} + \frac{t^6}{30} \tag{16}$$

$$x_1 = 2 \int_0^t \int_0^t \frac{d}{dt}(x_0) dt dt - 2 \int_0^t \int_0^t x_0 dt dt \tag{17}$$

$$= -0.2t^2 - 0.2t^3 - 0.0833t^4 + 0.05t^5 + 0.01944t^6 + 0.0051587t^7 - 0.0011904t^8 \tag{18}$$

$$x_2 = 2 \int_0^t \int_0^t \frac{d}{dt}(x_1) dt dt - 2 \int_0^t \int_0^t x_1 dt dt \tag{19}$$

$$= -0.133t^3 + 0.133t^4 + 0.0133t^5 + \dots + 0.000026455t^{10} \tag{20}$$

In a similar manner, the components x_k are calculated for $k = 3, 4, \dots$, but for brevity will not be listed. Considering (11), the approximate solution including eight terms is

$$x(t) \approx \varphi_8(t) = -0.4 - 0.6t - 0.2t^2 + 0.233t^3 + 0.3166t^4 + \dots - 1.26269 \times 10^{-15} t^{20} \tag{21}$$

Using this polynomial, the approximations of x values for $t = k/10, k = 0, 1, \dots, 10$ will be obtained as shown in Table 1, labeled by Standard "ADM".

Proposed Method:

In order to obtain the solution of equation (13), as a polynomial, we use the truncated Maclaurin series of its excitation force up to t^4 . To apply the modified technique introduced in section 3, we use the interpolated polynomial of the excitation force with the same order of t^4 . To do this, we evaluate the function $F(t) = e^{2t} \text{Sint}$ at points $t = 0, 0.25, 0.5, 0.75, 1$ to obtain five support points as follows

$$(0,0), (0.25,0.40790017007), (0.5, 1.30321372968), (0.75,3.05489298071), (1, 6.2176763123)$$

Using these support points we obtain a fourth order interpolated polynomial as

$$F(t) \approx P(t) = 0.96292421t + 2.3101200t^2 + 0.962913823t^3 + 1.981718264t^4 \tag{22}$$

Following the ADM procedure, and using (22), we obtain the solution of equation (13) as

$$x(t) \approx \varphi_8 = -0.4 - 0.6t - 0.19t^2 + 0.2271540t^3 + 0.339420t^4 + \dots - 2.50229 \times 10^{-15} t^{20} \tag{23}$$

The numerical results of this polynomial for $t = k/10, k = 0, 1, \dots, 10$ will be obtained as shown in Table 1, labeled by "modified ADM".

Example 2:

Consider the following version of the well-known Duffing's equation (Adomian, G., 1994)

$$x'' + 2x' + x + 8x^3 = e^{-3t}, \tag{24}$$

with the initial conditions $x(0) = 1/2$ and $x'(0) = -1/2$. The exact solution of this equation is $e^{-t}/2$, of which the numerical values for $0 \leq t \leq 1$ are presented in Table 2, labeled by "Exact".

i) Standard ADM:

Using the Maclaurin series of the excitation term e^{-3t} in the left side of equation (24) and adopting the terms up to t^4 we have

$$e^{-3t} = 1 - 3t = \frac{9t^2}{2} - \frac{9t^3}{2} + \frac{27t^4}{8} \tag{25}$$

Using (9) and (10) we obtain

$$x_0 = x(0) + x'(0)t + \int_0^t \int_0^t (1 - 3t = \frac{9t^2}{2} - \frac{9t^3}{2} + \frac{27t^4}{8}) dt dt \tag{26}$$

$$= \frac{1}{2} - \frac{t}{2} + \frac{t^2}{2} - \frac{t^3}{2} + \frac{3t^4}{8} - \frac{9t^5}{40} + \frac{9t^6}{80} \tag{27}$$

$$x_1 = 2 \int_0^t \int_0^t \frac{d}{dt} (x_0) dt dt - \int_0^t \int_0^t (x_0) dt dt - 8 \int_0^t \int_0^t A_0(x^3) dt dt \tag{28}$$

$$= -\frac{t^2}{4} + \frac{t^3}{4} - \frac{7t^4}{24} + \frac{3t^5}{8} - \frac{33t^6}{80} - \dots - 2.99 \times 10^{-5} t^{20} \tag{29}$$

$$x_2 = 2 \int_0^t \int_0^t \frac{d}{dt} (x_1) dt dt - 2 \int_0^t \int_0^t x_1 dt dt - 8 \int_0^t \int_0^t A_1(x^3) dt dt \tag{30}$$

$$= \frac{t^3}{6} + \frac{t^4}{48} - \frac{29t^5}{240} + \frac{139t^6}{720} - \frac{457t^7}{1680} + \dots + 8.11 \times 10^{-9} t^{34} \tag{31}$$

In a similar manner, the components x_k are calculated for $k = 3, 4, \dots$, but for brevity will not be listed. Considering (11), the approximate solution including six terms is

$$x(t) \approx \varphi_6(t) = \frac{1}{2} - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{12} + \frac{t^4}{48} - \frac{t^5}{240} + \frac{t^6}{1440} + \dots + 1.54 \times 10^{-19} t^{76} \tag{32}$$

Using this polynomial, the approximations of x values for $t = k/10, k = 0, 1, \dots, 10$ will be obtained as shown in Table 2, labeled by "Standard ADM".

II) Proposed Method:

As in example 1, by evaluating the excitation term $F(t) = e^{-3t}$ in $t = 0, 0.25, 0.5, 0.75, 1$, five support points will be obtained as follows

$$(0, 0), (0.25, 0.47236655274), (0.5, 0.22313016014), (0.75, 0.10539922456), (1, 0.04978706836)$$

which gives the interpolated polynomial as

$$F(t) \approx P(t) = 1 - 2.94069t + 3.97068t^2 - 2.80692t^3 + 0.826719t^4 \tag{33}$$

Following the ADM procedure, and using (33), we obtain the solution of equation (13) as

$$x(t) \approx \varphi_8(t) = \frac{1}{2} - \frac{t}{2} + 0.25t^5 - 0.073448t^3 - 0.0282194t^4 + 0.0699485t^5 - \dots - 2.9393 \times 10^{-26} t^{76} \tag{34}$$

The numerical results of this polynomial for $t = k/10, k = 0, 1, \dots, 10$ will be obtained as shown in Table 1, labeled by "modified ADM".

RESULTS AND DISCUSSIONS

In this section, we consider the results of standard ADM and our proposed method for the examples solved in the previous section. In order to verify the efficiency of the proposed method in comparison with exact solutions and standard ADM, we report the numerical results. Tables 1 and 2 show the results of the exact solution, standard ADM and proposed method applied to examples 1 and 2 for $0 \leq t \leq 1$, respectively. The second columns of the tables show the numerical results of the exact solution and the third and fourth columns show the results of standard ADM and the proposed method, respectively. In the fifth and sixth columns the absolute errors of each numerical method can be seen.

Table 1: Numerical results of methods applied to example 1

| Time | Exact | Standard ADM | Modified ADM | Standard Error | Modified Error |
|------|-------------|--------------|--------------|------------------------|-----------------------|
| 0 | 0.4 | 0.4 | 0.4 | 0 | 0 |
| 0.1 | -0.46173297 | -0.46173297 | -0.46173297 | 8.44×10^{-10} | 4.22×10^{-6} |
| 0.2 | -0.52555915 | -0.52555904 | -0.52555904 | 1.12×10^{-7} | 2.25×10^{-5} |
| 0.3 | -0.58860203 | -0.58860004 | -0.58860004 | 1.98×10^{-6} | 5.04×10^{-5} |
| 0.4 | -0.64662572 | -0.64661028 | -0.64661028 | 1.54×10^{-5} | 8.01×10^{-5} |
| 0.5 | -0.69364017 | -0.69356394 | -0.69356394 | 7.62×10^{-5} | 1.09×10^{-4} |
| 0.6 | -0.72143106 | -0.72114849 | -0.72114849 | 2.82×10^{-4} | 1.42×10^{-4} |
| 0.7 | -0.71900807 | -0.71814889 | -0.71814889 | 8.59×10^{-4} | 1.83×10^{-4} |
| 0.8 | -0.67196581 | -0.66970677 | -0.66970677 | 2.25×10^{-3} | 2.28×10^{-4} |
| 0.9 | -0.56175199 | -0.55643813 | -0.55643813 | 5.31×10^{-3} | 2.67×10^{-4} |
| 1 | -0.36483905 | -0.35339435 | -0.35339435 | 1.14×10^{-2} | 2.85×10^{-4} |

Table 2: Numerical results of the methods applied to example 2

| Time | Exact | Standard ADM | Modified ADM | Standard Error | Modified Error |
|------|----------|--------------|--------------|-----------------------|-----------------------|
| 0 | 0.5 | 0.5 | 0.5 | 0 | 0 |
| 1/10 | 0.452419 | 0.452419 | 0.452425 | 4.15×10^{-9} | 5.88×10^{-6} |
| 2/10 | 0.409365 | 0.409366 | 0.409392 | 4.87×10^{-7} | 2.67×10^{-5} |
| 3/10 | 0.370409 | 0.370417 | 0.370457 | 7.60×10^{-6} | 4.83×10^{-5} |
| 4/10 | 0.33516 | 0.335212 | 0.335212 | 5.17×10^{-5} | 5.24×10^{-5} |
| 5/10 | 0.303265 | 0.303489 | 0.303278 | 2.23×10^{-4} | 1.22×10^{-5} |
| 6/10 | 0.274406 | 0.275128 | 0.27426 | 7.22×10^{-4} | 1.46×10^{-4} |
| 7/10 | 0.248293 | 0.250204 | 0.247679 | 1.91×10^{-3} | 6.13×10^{-4} |
| 8/10 | 0.224664 | 0.229031 | 0.222867 | 4.36×10^{-3} | 1.79×10^{-3} |
| 9/10 | 0.203285 | 0.21219 | 0.198835 | 8.90×10^{-3} | 4.44×10^{-3} |
| 1 | 0.18394 | 0.200488 | 0.174119 | 1.65×10^{-2} | 9.82×10^{-3} |

The error of standard ADM for damped forced oscillator begins from the order 10^{-10} and ends by 10^{-2} . Applying interpolated polynomials instead of the Taylor expansion for $F(t) = e^{2t} \sin(t)$, the order of errors begins from 10^{-6} and ends by 10^{-4} .

The Taylor series of the excitation term is accurate about the expansion point and loses its accuracy as the independent variable goes away from the initial point. This affects the accuracy of the solutions obtained from ADM, which can be seen in the "Standard errors" columns of Tables 1 and 2. For example, in Table 1 we can see that the error of the standard method for $t = 0.1$ is of the order 10^{-10} , but away from this point, the Taylor series is not a good approximation of $F(t)$ and the error has an increasing regime and is unacceptable at the end of the interval. Applying the interpolated polynomial resolves this problem, as can be seen in the "Modified error" column.

Obviously, the interpolated polynomial coincides with the function at more than one point and so gives a more accurate approximation of the function. This causes the error of the modified method to have no considerable difference in the the lower and upper end of the interval as compared to standard ADM.

6 Conclusion:

In this work, we showed that the errors of the modified method presented in this paper are more uniform than standard ADM, i.e, the proposed method tends to yield more uniform convergence results for both linear and nonlinear equations. The numerical results show that while the standard ADM gives more accurate results at the lower end of the interval, the proposed approach tends to yield more uniform results.

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