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An Hyperidentity Concerning the Quasigroup Identities Isotopy Closure to Abelian Group

Kh. Shahbazpour

Department of Mathematical Science, Urmia University, Urmia, P.O.Box 165 IRAN

Abstract: V.D.Belousov in 1966 gave a characterization of quasigroups isotopic to abelian groups with four object variables (Belousov, V.D., 1966) (Theorem 9). In this paper we will show simple and smaller proof of theorem. Next, extend Belousov's theorem and characterized quasigroups isotopic to abelian groups by similar identities. Also, by these results we will find an hyperidentity such that it will characterized quasigroups isotopic to abelian groups.

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INTRODUCTION

An algebra $Q(\cdot)$ is called a quasi-group, if each of the following equations, $ax = b$ and $ya = b$, has only one solution for any $a, b \in Q$. A quasi-group with an identity element is called a loop.

The quasi-group concept has two interpretation. The first one is the combinatorial and in the form of a Latin square. The second is the geometrical one in the form of nets. In the language of algebraic systems we know that:

1. the class of all quasi-groups, which are isotopic to groups, forms a quasi-variety (See (Belavskaya, G., 2007)).
2. the class of all quasi-groups, which are isotopic to Abelian groups, forms a quasivariety(See (Belavskaya, G., 2007)).

In 1966, V.D. Belousov proved ((Belousov, V.D., 1966), also see (Belousov, V.D., 1965) and (Belousov, V.D., 1969)) that the classes of quasi-groups are also varieties. That is they are characterized by identities or are closed for homomorphisms, sub-algebras, and direct products.

If $Q(\cdot)$ is a quasi-group, then denoting the unique solution of the equation $ax = b$ by $x = a \backslash b$, and denoting the unique solution of the equation $ya = b$ by $y = b/a$, we get an algebra $Q(\cdot, \backslash, /)$ with the following identities:

$$x(x\backslash y)=y, \qquad x\backslash (x\cdot y)=y,$$
$$(y/x)\cdot x = y, \qquad (y\cdot x)/x = y.$$

V.D.Belousov in 1966 gave a characterization of quasigroups isotopic to abelian groups with four object variables (Belousov, V.D., 1966) (Theorem 9) (Also see (Belavskaya, G., 2007) for more results). In this paper we will show simple and smaller proof of theorem. Next, extend Belousov's theorem and characterized quasigroups isotopic to abelian groups by similar identities. Also, by these results we will find an hyperidentity such that it will characterized quasigroups isotopic to abelian groups.

The results can be used in the theory of quasi-groups, in the studies of the Latin squares, in the theory of nets, as well as in the invertible universal algebras.

§2. Hyperidentities:

An important extension of first-order logic is second-order logic. Second order formulae consist of the same logical symbols of &, V, ¬, →, ∃, ∀ of individual and functional (predicate) variables, that are used in

Corresponding Author: Kh. Shahbazpour, Department of Mathematical Science, Urmia University, Urmia P.O.Box 165 IRAN
E-mail: shahbazpour2003@hotmail.com

first order formulae. The difference is that in second order formulae, the quantifiers $\exists$, $\forall$ can be applied not only to individual variables, but also to functional (or to predicate) variables. Investigations on second order formulae go back to L. Henkin, A. I. Mal'tsev, A. Church, S. Kleene, A. Tarski (See (Chein, O., H.O. Pflugfelder, 1990), (Pflugfelder, H.O., 1990), (Willard, R., 2007)).

Starting with the 1960's the following second order formulae were studied in various domains of algebra and its applications

$$\forall X_1, \ldots, X_m \; \forall x_1, \ldots, x_n (w_1 = w_2), \tag{1}$$

$$\forall X_1,\ldots,X_k \; \exists X_{k+1},\ldots,X_m \forall x_1,\ldots,x_n(w_1 = w_2), \tag{2}$$

$$\exists x_1, \ldots, x_n \forall X_1, \ldots, X_m(w_1 = w_2), \tag{3}$$

$$\exists X_1,\ldots,X_k \forall X_{k+1},\ldots,X_m \forall x_1,\ldots,x_n(w_1 = w_2), \tag{4}$$

$$\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_t \forall X_{t+1}, \ldots, X_m \forall x_1, \ldots, x_n(w_1 = w_2), \tag{5}$$

where w_1, w_2 – are words (terms) in the functional variables $X_1, \ldots, X_m$ and the individual variables $x_1, \ldots, x_n$. The first formula is called a hyperidentity or $\forall(\forall)$-identity, the second (third,fourth,fifth) formula is called an $\forall\exists(\forall)$-identity ($(\exists)\forall$-identity, $\exists\forall(\forall)$-identity, $\forall\exists\forall(\forall)$-identity). Sometims the $\forall\exists(\forall)$-identity is called a generalized identity, the $(\exists)\forall$- identity is called a coidentity and $\exists\forall(\forall)$-identity is called a hybrid identity. The satisfiability of these second order formulae in an algebra $\mathfrak{A} = (Q; \Sigma)$ is understood by functional quantifiers $(\forall X_i)$ and $(\exists X_j)$, meaning: "for every value $X_i = A \in \Sigma$ of the corresponding arity" and "there exists a value $X_j = A \in \Sigma$ of the corresponding arity". It is assumed that such a replacement is possible, that is

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T\mathfrak{A}$$

where $|S|$ is the arity of S, and $T\mathfrak{A}$ is the arithmetic type of $\mathfrak{A}$.

Starting from 1954 the following-type second order formulae were studied in algebras of term functions of various classes of varieties

$$\exists X_1,\ldots,X_m \forall x_1,\ldots,x_n(w_1 = w_2), \tag{6}$$

which are called Mal'tsev (Mal'cev) conditions ((Chein, O., H.O. Pflugfelder, 1990)), and reduced to the hyperidentities of the class of term functions' (termal) algebras.

The investigation of hyperidentities is a comparatively new, actively developing field of pure and applied algebra. The concept of hyperidentity offers a high-level approach to algebraic questions, leading to new results, applications and problems. In particular the investigation of hyperidentities is useful from the view of new technologies too via optimization problems of block diagrams.

A class K of algebras of type τ is called a variety if τ is closed under the formation of homomorphic images, subalgebras, and direct products. Also, by Birkhoff's well-known theorem(See (Willard, R., 2007),Theorem 1.1), K is a variety if and only if there is set Σ of identities of type τ such that $K = mod\Sigma$, where $[mod\Sigma = \{A \in Alg(\tau) : A \models \Sigma\}]$.

Now We Consider Some Theorems That Are Useful for Our Main Results:

Theorem 2.1:

(*Belousov's theorem on four quasi-groups*). *If four quasi-groups* A_i ($i = 1, 2, 3, 4$) *defined on set Q, are connected by the general associative law* $A_1[A_2(x, y), z] = A_3[x, A_4(y, z)]$, *then all the* $Q(A_i)$ *are isotopic to one and the same group.*

Proof. See (Belousov, V.D., 1965) (Theorem 2.1):

In the following theorems, Belousov also proved that every quasi-group is isotopic to a group if and only if it satisfies some identity with five variables.

Theorem 2.2:

The quasi-group $Q(\cdot)$ is isotopic to a group if and only if the following identity holds in $Q(\cdot)$: $x\ \{y \setminus [(z/u)v]\} = \{[x(y \setminus z)]\ /u\}\ v$.

Proof. See (Belousov, V.D., 1966) (Theorem 8):

Theorem 2.3:

The quasi-group $Q(\cdot)$ is isotopic to an Abelian group if and only if the following identity holds in $Q(\cdot)$: $x\backslash[y(u\backslash v)] = u\backslash[y(x\backslash v)]$.

Proof. See (Belousov, V.D., 1966) (Theorem 9):

Theorem 2.4:

All operations in equivalence class K of non-cancelable balanced identities $w_1 = w_2$ *or* $\Phi_1(w) = \Phi_2(w)$ *are isotopic to one group, ƒ K contains more than two operation.*

Proof. See (Belousov, V.D., 1966) (Theorem 2):

This theorem is the generalization of theorem 2.1.

We will study the existence of all four-variable balanced identities that characterize quasi-groups isotopic to abelian groups. Such identities are called linear equations or balanced identities in the sense of Prof. V. D. Belousov(See (Belousov, V.D., 1966) and (Belousov, V.D., 1969)).

Definition 2.1:

Let $A = \langle Q; \Sigma \rangle$ *and* $A' = \langle Q'; \Sigma' \rangle$ *are binary algebras, that is, algebras with binary operations. An algebra, A, is called isotopic to algebra A', ƒ there exist permutations* $\alpha, \beta, \gamma : Q \to Q'$ *and bijection* $\tilde{\psi} : \Sigma \to \Sigma 0$ *such that for any operator* $A \in \Sigma$ *and any* $x, y \in Q$ *we have*

$$\alpha A(x,y) = \left[\tilde{\Psi} A\right](\beta x, \gamma y) \qquad (7)$$

In addition, if

$$x.y = \gamma^{-1}(\alpha x \circ \beta y) \qquad (8)$$

and $x \cdot y = z$ then one can obtain $x = z/y$ and $y = x \setminus z$ therefore, from (8) we can write

$$\gamma z = \alpha x \circ \beta y \Rightarrow \alpha x = \gamma z \circ \overline{\beta y} \Rightarrow x = \alpha^{-1}(\gamma z \circ \overline{\beta y})$$

also $\beta y = \overline{\alpha x} \circ \gamma z \Rightarrow y = \beta^{-1}(\overline{\alpha x} \circ \gamma z)$. where $\alpha \circ \bar{\alpha} = \beta \circ \bar{\beta} = e(\bar{\alpha} x = (ax)^{-1})$ identity map of group.

Note. For simplicity, we put λx for βx, δx for αx and therefore, we use the following notations:

$$x.y = \gamma^{-1}(\alpha x \circ \beta y),$$

$$x/y = \alpha^{-1}(\gamma x \circ \delta y),$$

$$x/y = \beta^{-1}(\lambda x \circ \gamma y).$$

Also, we note that if quasi-group $Q(\cdot, \backslash, /)$ is isotopic to group $Q(\circ)$ then each of quasi-groups $Q(\cdot)$, $Q(\backslash)$ and $Q(/)$ is isotopic to the same group, $Q(\circ)$.

Question:

Precisely which balanced identities are satisfied by an arbitrary quasi-group iff the latter is isotopic to a group (or abelian group)?

The idea of balanced identities goes back to A. Sade(See (Belousov, V.D., 1969) and (Chein, O., H.O. Pflugfelder, 1990)). The equation v = w is called balanced if each object variable has exactly one appearance on the left-hand side and one appearance on the right-hand side of the equation.

Since all functional variables in the general equation are chosen from the set {·, \, /}, we prefer to denote them by Ai's. First, we present all general identities:

1. $A_1\left[A_2\left[A_3(x,y)z\right],u\right] = A_4\left[x, A_5\left[y, A_6(z,u)\right]\right]$
2. $A_1\left[A_2\left[A_3(x,y)z\right],u\right] = A_4\left[x, A_5(x,y), A_6(z,u)\right]$
3. $A_1\left[A_2\left[A_3(x,y)z\right],u\right] = A_4\left[A_5\left[x, A_6(y,u)\right],u\right]$
4. $A_1\left[A_2\left[A_3(x,y)z\right],u\right] = A_4\left[x, A_5\left[A_6(y,u)\right],u\right]$
5. $A_1\left[A_2(x,y)A_3(x,y)z\right] = A_4\left[A_5\left[x, A_6(y,u)\right],u\right]$
6. $A_1\left[A_2(x,y)A_3(x,y)z\right] = A_4\left[x, A_5\left[y, A_6(z,u)\right]\right]$
7. $A_1\left[A_2(x,y)A_3(x,y)z\right] = A_4\left[x, A_5\left[A_6(y,u)\right]\right]$
8. $A_1\left[x, A_2\left[y, A_3(z,u)\right]\right] = A_4\left[A_5\left[x, A_6(y,z)\right],u\right]$
9. $A_1\left[x, A_2\left[y, A_3(z,u)\right]\right] = A_4\left[x, A_5\left[A_6(y,z),u\right]\right]$
10. $A_1\left[A_2\left[x, A_3(y,z)\right],u\right] = A_4\left[x, A_5\left[A_6(y,z),u\right]\right]$
11. $A_1\left[x, A_2\left[A_3(y,z),u\right]\right] = A_4\left[x, A_5\left[A_6(y,z),u\right]\right]$
12. $A_1\left[A_2\left[x, A_3(y,z)\right],u\right] = A_4\left[A_5\left[x, A_6(y,z)\right],u\right]$
13. $A_1\left[A_2\left[x, A_3(z,u)\right]\right] = A_4\left[A_5\left[x, A_6(z,u)\right]\right]$
14. $A_1\left[A_2(x,y), A_3(z,u)\right] = A_4\left[A_5\left[(x,y).A_6(z,u)\right]\right]$
15. $A_1\left[A_2\left[A_3(x,y),z\right]u\right] = A_4\left[A_5\left[A_6(x,y),z\right],u\right]$

Also for each case, and for any A_i, we have a three-operator from: {·, \ , /}. So we have 729 identities to be investigated for each case.

§3. Main results:

We will show that as generalization of theorem 2.3, we can present the following theorems for quasi-groups isotopic to some abelian groups (See (Shahbazpour, Kh., 2007) and (Shahbazpour, Kh., 2005) for more details). Also, note that this theorem proved in 40 pages by Prof. V.D. Belousov ((Belousov, V.D., 1966)). We will show small prove of our theorems, by useing Belousov's theorem about four quasigroups(Theorem 2.1).

Theorem 3.5:

The quasi-group Q(·, \, /) is isotopic to an abelian group if and only if in Q(·, \, /) the following identity holds:

(a) x/ [(y/z) \ u] = y/ [(x/z) \ u].

Proof (*a*). Let identity (*a*) is satisfied in quasi-group *Q*(·, \, /). By *x* = *a* (a is an arbitrary fixed element), we can write identity (a) in the form:

$$B\left[A(x,z),u\right] = A\left[y,c(z,u)\right]. \tag{9}$$

Where we consider $A(x, y) = x/y$, $B(x, y) = a/(x\backslash y)$, $c(x, y) = (a/x)\backslash y$. Hence, by theorem 2.1, it is obvious that $Q(A)$, $Q(B)$, $Q(C)$ are quasi-groups isotopic to one and the same group. Next, from $A(x, y) = x/y$ we consider the isotopy $(/) \sim_{iso}. (\circ)$ such that $\gamma(x/y) = \alpha x \circ \beta y$. (10)
by relation, we have

$$x/y=\gamma^{-1}(\alpha x \circ \beta y) \text{ and } x\backslash y=\beta^{-1}(\alpha y \circ \gamma x), \tag{11}$$

where α^{-} denotes the inverse of α in group $Q(\circ)$.

Therefore, substituting (10), (11) in (a) and calculating two sides, we obtain:

$$\gamma^{-1}(\alpha x \circ \overline{\alpha u} \circ \alpha y \circ \beta z) = \gamma^{-1}(\alpha y \circ \overline{\alpha u} \circ \alpha x \circ \beta z) \tag{12}$$

Equivalently

$$\alpha x \circ \overline{\alpha u} \circ \alpha y = \alpha y \circ \overline{\alpha u} \circ \alpha x \tag{13}$$

New replace αx, αy and $\overline{\alpha u}$ by x, y, u respectively, hence:

$$x \circ u \circ y = y \circ u \circ x. \tag{14}$$

It follows that group $Q(\circ)$ is abelian.

Conversely: let the quasi-group $Q(\cdot, \backslash, /)$ be isotopic to abelian group $Q(\circ)$, with isotopy $\gamma(x/y) = \alpha x \circ \beta y$. Then, we have $x/y = \gamma^{-1}(\alpha x \circ \beta y)$ and $x \backslash y = \beta^{-1} (\alpha y \circ \gamma x)$. Now, because the group $Q(\circ)$ is abelian. For each of $x, y, z, u \in Q$, we can write:

$$x \circ u \circ y \circ z = y \circ u \circ x \circ z. \tag{I}$$

By substituting αx, αy, βz and αu for x, y, z, u respectively, and acting by γ^{-1} on the two sides, we obtain

$$\gamma^{-1} (\alpha x \sim \alpha u \sim \alpha y \sim \beta z) = \gamma^{-1} (\alpha y \sim \alpha u \sim \alpha x \sim \beta z). \tag{II}$$

Hence we obtain the identity (a) from (II) .

Theorem 3.6:

Quasi-group $Q(\cdot, \backslash, /)$ is isotopic to an abelian group if and only if in $Q(\cdot, \backslash, /)$ the following identities holds:

$(b)\ x[y\backslash(z.u)] = x/[z\backslash(y.u)].$

$(c)\ [(x/y).z]/u = [(x/u).z]/y$

$(d)\ (x.y)/(z\backslash u) = (z.y)/(x\backslash u).$

Formulas (b), (c) and (d), by let the isotopy $x/y = \gamma^{-1}(\alpha x \circ \beta y)$, have similar proofs 3.5.

Theorem 3.7:

Quasi-group $Q(\cdot, \backslash, /)$ is isotopic to an abelian group if and only if in $Q(\cdot, \backslash, /)$ the following identities holds:

(a) $x\cdot [y\backslash(z\cdot u)]=z\cdot [y\backslash(x\cdot u)]$

(b) $[(x \cdot y)/z] \cdot u = [(x \cdot u)/z] \cdot y (x/y) \cdot (z\backslash u) = (u/y) \cdot (z\backslash x)$
$x \cdot [y/(z \cdot u)] = z \cdot [y/(x \cdot u)]$

Proof. by proof of the previous theorem 3.5, it is enough to let isotopy $(\cdot) \sim_{iso}. (\circ)$, such that $\gamma(x \cdot y) = \alpha x \circ \beta y$.

Theorem 3.8:
Quasi-group $Q(\cdot, \backslash, /)$ is isotopic to an abelian group if and only if in $Q(\cdot, \backslash, /)$ the following identities holds:

$(x \cdot y)/(z\backslash u)=(z \cdot y)/(x\backslash u)$ (f)$x\backslash[y \cdot (z\backslash u)] =z\backslash[y \cdot (x\backslash u)]$ (g)$x\backslash[y/(z\backslash u)] =z\backslash[y/(x\backslash u)]$

Proof. It is sufficient to consider isotopy $(\cdot) \sim_{iso}. (\circ)$ and the proof follows from the previous theorem 3.5.

Theorem 3.9:
Quasi-group $Q(\cdot, \backslash, /)$ is isotopic to an abelian group if and only if in $Q(\cdot, \backslash, /)$ the following identities holds:

$x/[y \cdot (z/u)] = z/[y \cdot (x/u)]$
$x/[y \backslash (z/u)] = z/[y \backslash (x/u)]$.

Proof. It is sufficient to consider isotopy $(\cdot) \sim_{iso}. (\circ)$ and the proof follows from the previous theorem 3.5.

Main theorem:
Theorem 3.10:
The class of all quasigroups isotopic to abelian groups can be characterized with the following hyperidentity:

$A_1[x,A_2(y,A_1(z,u))] = A_1[z,A_2(y,A_1(x,u))]$.

Therefor we can characterize all quasigroup identities with four object variables isotopy closure to abelian groups.
Proof. It is sufficient to consider above theorems 3.5, 3.6, 3.7, 3.8, 3.9.

§4. Conclusion:
By methods of (Shahbazpour, Kh., 2007), (Shahbazpour, Kh., 2005), we have simple proof of Belousov's theorem ((Belousov, V.D., 1966), Theorem 2). He proved this theorem in 40 pages. Then by classification of identities for quasigroups isotopic to abelian groups, we proved that:
The class of all quasigroups isotopic to abelian groups characterized with four object variables, by the following hyperidentity:

$A_1[x,A_2(y,A_1(z,u))] = A_1[z,A_2(y,A_1(x,u))]$.

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