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SBM Model with Fuzzy Input-output Levels in DEA

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Abstract: In this paper a model for fuzzy SBM of efficiency in DEA is presented based on α-cuts of the fuzzy number parameters to convert fuzzy linear programming problem to an interval programming problem. Then, by introducing a variable in each of the obtained intervals and applying suitable substitutions, a linear programming problem would be achieved. To demonstrate the concept, numerical examples are solved and solutions are compared with Jahanshahloo et al. (Applied Mathematics and Computation 156, 175-187).

Keywords: Data envelopment analysis, Membership function, Fuzzy linear programming, Slack-Based measure of efficiency

INTRODUCTION

Efficiency analysis of Decision Making Units (DMUs) such as the branches of a bank, schools, post offices, etc. is extensively being evaluated by Data Envelopment Analysis (DEA) (Emrouznejad *et al.* 2008). Evaluating the efficiency of DMUs by traditional DEA models requires crisp input/output data. Reducing complex real-world systems into precise mathematical model always is the main trend in science. However, real-world situations are often not as deterministic as people recognized. To deal with imprecision/uncertainty, the fuzzy set theory is useful. For imprecise information, some authors have developed Fuzzy DEA (see. Kao and Liu, (2000). Jahanshahloo, *et al.,* (2004). Guo and (2009) Wen and Li (2009). and Saati *et al.* (2002). The methods in Fuzzy DEA, for defuzzification, use one of the ranking algorithms. Recently, Jahanshahloo *et al.* (2004). proposed a membership function to determine the relation among two triangular fuzzy numbers and used it to solve the Slack-Based Measure (SBM) model with fuzzy inputs/outputs. They defined a membership function based on the Carlsson and Fuller's (2001). possibilistic mean value to determine the relation among two triangular fuzzy numbers. However, the proposed membership function in Jahanshahloo *et al.* (2004). is problematic sometimes and the resulted model is a multi objective nonlinear programming problem (see Saati and Memariani, 2007).

In this paper, using the suggested method in Saati *et al.* (2002). we develop a linear programming problem for fuzzy SBM model. The proposed model is based on the α-cuts of the objective and constraints of fuzzy SBM model. In most α-cut based methods, the resulting model is solved by comparing two intervals, i.e., interval of LHS and interval of RHS of each equality/inequality constraint. This comparison either oversimplifies the model or makes it difficult to solve. The proposed approach presumes that the solution lies in the interval and defines suitable variables for this solution. The substitutions of these variables make the model non-linear. By further suitable substitutions the model is linearized. Hence, by solving a linear programming problem for a given α-cut, it is possible to generate a reliable and robust solution for possibilistic mathematical programming problems in general and fuzzy SBM model in particular.

The paper is organized as follow: The suggested method to solve fuzzy SBM model is presented in section 2. To demonstrate the concept, numerical examples are given in section 3 and the results are compared with the results of Jahanshahloo *et al.,* (2004). Section 4 closes with conclusion.

I.SBM model with fuzzy input and output data:

Assume that there are n DMUs to be evaluated. Each DMU consumes varying amounts of m different fuzzy inputs to produce s different fuzzy outputs. Specifically, DMU_j ($j = 1,...,n$) consumes amounts x_{ij} ($i=1,...,m$) of inputs and produces amounts y_{rj} ($r=1,...,s$) of outputs. In the model formulation, x_{ip} ($i=1,...,m$) and y_{rp} ($r=1,.m,s$) denote, respectively, the input and output values for DMU_p.
The proposed SBM model by Tone 2001. is as follows:

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$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} \\
s.t: \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 && j = 1,\dots,n, \\
& (m x_{ip}) v_i \ge 1 && i = 1,\dots,m, \\
& \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} + 1 \le s u_r y_{rp} && r = 1,\dots,s, \\
& u_r \ge 0,\ v_i \ge 0 && r = 1,\dots,s,\ i = 1,\dots,m.
\end{aligned} \tag{1}
$$

In this model, m and s are the number of inputs and outputs, respectively and, u_r ($r=1,\dots,s$) and v_i ($i=1,\dots,m$) are variables. SBM model with fuzzy data can be stated as follows:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rp} - \sum_{i=1}^{m} v_i \tilde{x}_{ip} \\
s.t: \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le \tilde{0} && j = 1,\dots,n, \\
& (m \tilde{x}_{ip}) v_i \ge \tilde{1} && i = 1,\dots,m, \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rp} - \sum_{i=1}^{m} v_i \tilde{x}_{ip} + \tilde{1} \le s u_r \tilde{y}_{rp} && r = 1,\dots,s, \\
& u_r \ge 0, v_i \ge 0 && r = 1,\dots,s, i = 1,\dots,m.
\end{aligned} \tag{2}
$$

where, '~' indicates the fuzziness. Let, $\tilde{x}_{ij} = (x^m_{ij}, x^l_{ij}, x^u_{ij})$, $\tilde{y}_{rj} = (y^m_{rj}, y^l_{rj}, y^u_{rj})$, $\tilde{1} = (1, 1^l, 1^u)$ $(1^l \le 1 \le 1^u)$ and $\tilde{0} = (0, 0^l, 0^u)$ $(0^l \le 0 \le 0^u)$.

To solve (2), we apply the proposed method in Saati, *et al.,* (2002). By introducing α-cuts of objective function and constraints, the following problem would be obtained:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r [\alpha y^m_{rp} + (1-\alpha) y^l_{rp}, \alpha y^m_{rp} + (1-\alpha) y^u_{rp}] \\
& - \sum_{i=1}^{m} v_i [\alpha x^m_{ip} + (1-\alpha) x^l_{ip}, \alpha x^m_{ip} + (1-\alpha) x^u_{ip}] \\
s.t: \quad & \sum_{r=1}^{s} u_r [\alpha y^m_{rj} + (1-\alpha) y^l_{rj}, \alpha y^m_{rj} + (1-\alpha) y^u_{rj}] \\
& - \sum_{i=1}^{m} v_i [\alpha x^m_{ij} + (1-\alpha) x^l_{ij}, \alpha x^m_{ij} + (1-\alpha) x^u_{ij}] \le [(1-\alpha) 0^l, (1-\alpha) 0^u] && \forall j, \\
& m[\alpha x^m_{ip} + (1-\alpha) x^l_{ip}, \alpha x^m_{ip} + (1-\alpha) x^u_{ip}] v_i \ge [\alpha + (1-\alpha) 1^l, \alpha + (1-\alpha) 1^u] && \forall i, \\
& \sum_{r=1}^{s} u_r [\alpha y^m_{rp} + (1-\alpha) y^l_{rp}, \alpha y^m_{rp} + (1-\alpha) y^u_{rp}] \\
& - \sum_{i=1}^{m} v_i [\alpha x^m_{ip} + (1-\alpha) x^l_{ip}, \alpha x^m_{ip} + (1-\alpha) x^u_{ip}] + [\alpha + (1-\alpha) 1^l, \alpha + (1-\alpha) 1^u] \\
& \qquad \le s u_r [\alpha y^m_{rp} + (1-\alpha) y^l_{rp}, \alpha y^m_{rp} + (1-\alpha) y^u_{rp}] && \forall r, \\
& \qquad u_r \ge 0, v_i \ge 0 && \forall r, i.
\end{aligned} \tag{3}
$$

To linearize the interval programming problem (3), following substitutions are introduced:

$$p \in [(1-\alpha)0^l, (1-\alpha)0^u]$$
$$q \in [\alpha + (1-\alpha)1^l, \alpha + (1-\alpha)1^u]$$
$$\hat{x}_{ij} \in [\alpha x_{ij}^m + (1-\alpha)x_{ij}^l, \alpha x_{ij}^m + (1-\alpha)x_{ij}^u] \qquad \forall i, j,$$
$$\hat{y}_{rj} \in [\alpha y_{rj}^m + (1-\alpha)y_{rj}^l, \alpha y_{rj}^m + (1-\alpha)y_{rj}^u] \qquad \forall r, j.$$

By substituting the new variables, (3) can be written as follows:

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \hat{y}_{rp} - \sum_{i=1}^{m} v_i \hat{x}_{ip} \\
st: \quad & \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \le p && \forall j, \\
& m\hat{x}_{ip} v_i \ge q && \forall i, \\
& \sum_{r=1}^{s} u_r \hat{y}_{rp} - \sum_{i=1}^{m} v_i \hat{x}_{ip} + q \le s u_r \hat{y}_{rp} && \forall r, \\
& (1-\alpha)0^l \le p \le (1-\alpha)0^u \\
& \alpha + (1-\alpha)1^l \le q \le \alpha + (1-\alpha)1^u \\
& \alpha x_{ij}^m + (1-\alpha)x_{ij}^l \le \hat{x}_{ij} \le \alpha x_{ij}^m + (1-\alpha)x_{ij}^u && \forall i, j, \\
& \alpha y_{rj}^m + (1-\alpha)y_{rj}^l \le \hat{y}_{rj} \le \alpha y_{rj}^m + (1-\alpha)y_{rj}^u && \forall r, j, \\
& q, u_r, v_i, \hat{x}_{ij}, \hat{y}_{rj} \ge 0 && \forall r, i, j.
\end{aligned} \tag{4}$$

In order to linearize this model, following substitutions are performed:

$$\bar{x}_{ij} = v_i \hat{x}_{ij}, \qquad \bar{y}_{rj} = u_r \hat{y}_{rj} \qquad \forall i, r, j.$$

By these substitutions, (4) will become a linear problem as follows:

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} \bar{y}_{rp} - \sum_{i=1}^{m} \bar{x}_{ip} \\
st: \quad & \sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} \le p && \forall j, \\
& m\bar{x}_{ip} \ge q && \forall i, \\
& \sum_{r=1}^{s} \bar{y}_{rp} - \sum_{i=1}^{m} \bar{x}_{ip} + q \le s\bar{y}_{rp} && \forall r, \\
& (1-\alpha)0^l \le p \le (1-\alpha)0^u \\
& \alpha + (1-\alpha)1^l \le q \le \alpha + (1-\alpha)1^u \\
& (\alpha x_{ij}^m + (1-\alpha)x_{ij}^l)v_i \le \bar{x}_{ij} \le (\alpha x_{ij}^m + (1-\alpha)x_{ij}^u)v_i && \forall i, j, \\
& (\alpha y_{rj}^m + (1-\alpha)y_{rj}^l)u_r \le \bar{y}_{rj} \le (\alpha y_{rj}^m + (1-\alpha)y_{rj}^u)u_r && \forall r, j, \\
& q, u_r, v_i, \bar{x}_{ij}, \bar{y}_{rj} \ge 0 && \forall r, i, j.
\end{aligned} \tag{5}$$

Simply, by considering the structure of (5), it can be reduced to the following linear programming problem:

$$\begin{aligned}
\max\quad & \sum_{r=1}^{s}\bar{y}_{rp}-\sum_{i=1}^{m}\bar{x}_{ip} \\
s.t:\quad & \sum_{r=1}^{s}\bar{y}_{rj}-\sum_{i=1}^{m}\bar{x}_{ij}\le(1-\alpha)0^{u} && \forall j,\\
& m\bar{x}_{ip}\ge\alpha+(1-\alpha)1^{l} && \forall i,\\
& \sum_{r=1}^{s}\bar{y}_{rp}-\sum_{i=1}^{m}\bar{x}_{ip}+\alpha+(1-\alpha)1^{l}\le s\bar{y}_{rp} && \forall r,\\
& (\alpha x_{ij}^{m}+(1-\alpha)x_{ij}^{l})v_i\le\bar{x}_{ij}\le(\alpha x_{ij}^{m}+(1-\alpha)x_{ij}^{u})v_i && \forall i,j,\\
& (\alpha y_{rj}^{m}+(1-\alpha)y_{rj}^{l})u_r\le\bar{y}_{rj}\le(\alpha y_{rj}^{m}+(1-\alpha)y_{rj}^{u})u_r && \forall r,j,\\
& u_r,v_i,\bar{x}_{ij},\bar{y}_{rj}\ge 0 && \forall r,i,j.
\end{aligned}\tag{6}$$

Definition: DMU_p is said to be efficient if the optimal value of (6) is zero.
Consider the first constraint of (6):

$$\sum_{r=1}^{s}\bar{y}_{rj}-\sum_{i=1}^{m}\bar{x}_{ij}\le(1-\alpha)0^{u}\qquad\forall j,$$

If $0^u > 0$ then, some of the DMUs may obtain efficiency greater than 1. So, it has to be equal to zero. Similarly, it can be obtained that $1^l=1$. Therefore, (6) can be written as:

$$\begin{aligned}
\max\quad & \sum_{r=1}^{s}\bar{y}_{rp}-\sum_{i=1}^{m}\bar{x}_{ip} \\
s.t:\quad & \sum_{r=1}^{s}\bar{y}_{rj}-\sum_{i=1}^{m}\bar{x}_{ij}\le 0 && \forall j,\\
& m\bar{x}_{ip}\ge 1 && \forall i,\\
& \sum_{r=1}^{s}\bar{y}_{rp}-\sum_{i=1}^{m}\bar{x}_{ip}+1\le s\bar{y}_{rp} && \forall r,\\
& (\alpha x_{ij}^{m}+(1-\alpha)x_{ij}^{l})v_i\le\bar{x}_{ij}\le(\alpha x_{ij}^{m}+(1-\alpha)x_{ij}^{u})v_i && \forall i,j,\\
& (\alpha y_{rj}^{m}+(1-\alpha)y_{rj}^{l})u_r\le\bar{y}_{rj}\le(\alpha y_{rj}^{m}+(1-\alpha)y_{rj}^{u})u_r && \forall r,j,\\
& u_r,v_i,\bar{x}_{ij},\bar{y}_{rj}\ge 0 && \forall r,i,j.
\end{aligned}\tag{7}$$

which is equivalent to a parametric programming, while $\alpha\in[0,1]$ is a parameter. Thus, the fuzzy SBM model (2) is equivalent to crisp parametric linear programming problem (7).

Theorem 1. If $x_{ij}^{l}=x_{ij}^{u}=x_{ij}^{m}\ (\forall i,j)$ and $y_{rj}^{l}=y_{rj}^{u}=y_{r}^{m}\ (\forall r,j)$ then, (7) is equivalent to the SBM model (1).

3. Numerical example:

Consider the introduced numerical example by Guo and Tanaka (2001). with fuzzy single-input and single-output. Data are listed in Table 1.

Table 1: Data for numerical example of Guo and Tanaka(2001).

DMUs	*A*	*B*	*C*	*D*	*E*
Input	(2.0,1.5,2.5)	(3.0,2.5,3.5)	(3.0,2.4,3.6)	(5.0,4.0,6.0)	(5.0,4.5,5.5)
Output	(1.0,0.7,1.3)	(3.0,2.3,3.7)	(2.0,1.6,2.4)	(4.0,3.0,5.0)	(2.0,1.8,2.2)

The efficiencies of these DMUs by using the fuzzy CCR model of (Saati *et al.,* 2002). are listed in Table 2 by different *a*. As seen, *B* is efficient for any $\alpha\in[0,1]$ an *E* is not.

Table 2: Efficiencies by fuzzy CCR model. of (Saati *et al.,* 2002).

DMUs	*A*	*B*	*C*	*D*	*E*
α=0					
*u**	0.77	0.43	0.62	0.33	0.34
*v**	0.51	0.29	0.42	0.22	0.22
Efficiency	1.00	1.00	1.00	1.00	0.74
α=0.5					
*u**	0.70	0.38	0.45	0.27	0.26
*v**	0.57	0.31	0.37	0.22	0.21
Efficiency	0.81	1.00	1.00	1.00	0.54
α=0.75					
*u**	0.59	0.35	0.39	0.23	0.23
*v**	0.53	0.32	0.35	0.21	0.21
Efficiency	0.63	1.00	0.82	0.99	0.47
α=1					
*u**	0.50	0.33	0.33	0.20	0.20
*v**	0.50	0.33	0.33	0.20	0.20
Efficiency	0.50	1.00	0.67	0.80	0.40

Table 3: Results of Jahanshahloo *et al,*(2004).

DMUs	*A*	*B*	*C*	*D*	*E*
λ*	0.82	1.00	0.91	0.96	0.74
*u**	0.51	0.33	0.34	0.20	0.21
*v**	0.48	0.33	0.33	0.20	0.19
Center of profit	-0.45	0.00	-0.31	-0.20	-0..55

Using the suggested method in Jahanshahloo *et al.* (2004), the approximate solutions (with ε_1=0.01) are obtained. The results are presented in Table 3.

Using (7) for these five DMUs, following results by different $\alpha \in \{0,0.5,0.75,1\}$ are obtained.

The results of the proposed method are same as the Jahanshahloo *et al.* (2004). However, in contrast to Jahanshahloo *et al.* (2004). these results are obtained by solving some linear programming problems and are not approximate solutions.

Table 4: Results of suggested method

DMUs	*A*	*B*	*C*	*D*	*E*
α=0					
*u**	0.77	0.43	0.50	0.20	0.34
*v**	0.67	0.38	0.42	0.17	0.22
Opt. val. of (7)	0.00	0.00	0.00	0.00	-0.26
α=0.5					
*u**	0.70	0.38	0.45	0.22	0.26
*v**	0.57	0.31	0.37	0.18	0.21
Opt. val. of (7)	-0.19	0.00	0.00	0.00	-0.46
α=0.75					
*u**	0.59	0.35	0.39	0.23	0.23
*v**	0.53	0.32	0.35	0.21	0.21
Opt. val. of (7)	-0.37	0.00	-0.18	-0.01	-0.53
α=1					
*u**	0.50	0.33	0.33	0.20	0.20
*v**	0.50	0.33	0.33	0.20	0.20
Opt. val. of (7)	-0.50	0.00	-0.33	-0.20	-0.60

The suggested method, is also feasible after imposing the weights restrictions $1 \le u <,4v$ and $0 \le v \le 1$ (or $1 \le u$ and $0 \le v \le 1$). Considering these constraints in (4) and α=0.5, yield the results on Table 5.

As seen in Table 5, in contrast to which Jahanshahloo *et al.* (2004) claimed, our method is feasible after imposing the above restrictions on weights.

Table 5: Results of bounded weights.

DMUs	*A*	*B*	*C*	*D*	*E*
*u**=1.00					
*v**=0.82					
Opt. val. of (7)	-0.28	0.00	0.00	0.00	-1.77

Conclusion:

The DMUs lying on the weak efficient frontier are demonstrated efficient by some of DEA models. But, actually these DMUs are not efficient. SBM for efficiency analysis is an approach to distinguish these DMUs and evaluate them according to their distance from the nearest efficient extreme point. Recently, there has been an attempt to adopt the concept of SBM in fuzzy DEA. In this paper, we criticize the method suggested by Jahanshahloo *et al.* (2004). and develop the fuzzy SBM in such a way that by suitable substitution of variables the solution is obtained with different α-cuts. The benefit of our method in comparison with Jahanshahloo *et al.,* (2004) is that our models are linear and the solutions are exact.

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