

A Variational Iteration Method for Solving the Nonlinear Klein-Gordon Equation

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Abstract: In this paper, variational iteration method (VIM) is presented as an alternative method for solving the nonlinear Klein-Gordon equation. The method is demonstrated by several examples. A new technique for choosing the initial approximation of VIM is presented. Comparisons with the exact solutions reveal that VIM is very effective and convenient.

Key words: Nonlinear PDE; Variational iteration method; Klein-Gordon equation; lagrange multiplier

INTRODUCTION

It is well known that many phenomena in scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can be modelled by nonlinear partial differential equations. The non-linear models of real-life problems are still difficult to solve either numerically or theoretically. A broad class of analytical solutions methods and numerical solutions methods were used to handle these problems (Wang, 1988; Jeffery and Mohamad, 1991; Wadati, 1972).

Our attention will focus on the nonlinear Klein-Gordon equation of the form

$$u''(x,t) - u_{xx}(x,t) + bu(x,t) + g(u(x,t)) = f(x,t), \quad (1)$$

$$u(x,0) = a_0(x), \quad u_t(x,0) = a_1(x), \quad (2)$$

where b is a real number, g is a given nonlinear function, and f is a known function.

The Klein-Gordon equation is one of the more important mathematical models in quantum mechanics (Whitham, 1974; Zauderer, 1983). The equation has attracted much attention in studying solitons and condensed matter physics (Caudery *et al.*, 1975), in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations (Dodd *et al.*, 1982). With reference to the numerical solution for this problem we can see many published papers. Many authors (Deeba and Khuri, 1996; El-Sayed, 2003; Kaya and El-Sayed, 2004; Wazwaz, 2006) used Adomian's decomposition method for solving linear and nonlinear Klein-Gordon equations. Inc (Inc, 2006) investigate the special exact solutions of the modified nonlinearly Klein-Gordon-type equations by using some ansatze, and obtained new soliton solutions with compact support and solitary pattern solutions having infinite slopes or cusps, solitary wave and periodic solutions. Wazwaz (Wazwaz, 2006) studied the nonlinear Klein-Gordon equations with power law nonlinearities, used the tanh method for analytic treatment for these equations. The analysis led to travelling wave solutions with compactons, solitons, solitary patterns and periodic structures.

Another powerful analytical method, called the variational iteration method (VIM), was first envisioned by He (He, 1998) (He, 1998; 1999; 2000; He, *et al.*, 2004; 2006; 2006; 2006). VIM has successfully been applied to many situations. For example, He (1998) solved the classical Blasius' equation using VIM. He (1999) used VIM to give approximate solutions for some well-known non-linear problems. He (2000) used VIM to solve autonomous ordinary differential systems. He (2006) solved strongly nonlinear equations using VIM. Soliman (2005) applied VIM to solve the KdV-Burger's and Lax's seventh-order KdV equations. VIM was employed for solving non-linear coagulation problem with mass loss by Abulwafa *et al.* (2005). Momani *et al.* (2006) applied VIM to the Helmholtz equation. VIM has been applied for solving nonlinear differential equations of fractional order by Odibat *et al.* (2006). Bildik *et al.* (2006) used VIM to solve different types of nonlinear partial differential equations. Wazwaz (2006) used VIM to determine rational solutions for the

KdV, K (2,2), Burgers' and cubic Boussinesq equations. Wazwaz, (2006) presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer *et al.* (2006) introduced a modification of VIM. Abbasbandy (2006) solved the quadratic Riccati differential equation by VIM incorporating Adomian's polynomials. Junfeng (2006) applied VIM to solve singular two-point boundary value problems. Batiha *et al.* (2006) applied VIM to solve the generalized Huxley equation and in (Batiha *et al.*, 2006) they employed VIM to the generalized Burgers-Huxley equation.

The purpose of this paper is to apply VIM to find the approximate analytical solution of the linear and nonlinear Klein-Gordon equation(1), show the effective of the new technique of choosing the initial approximation. Comparisons with the exact solution shall be performed graphically.

Variational Iteration Method:

VIM is based on the general Lagrange's multiplier method (Inokuti *et al.*, 1978). The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution (He, 2006).

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to VIM (He, 1999; 2000; He, *et al.*, 2004; 2006; 2006; 2006), we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau) [Lu_n(\tau) + Nu_n(\tau) - g(\tau)] d\tau, \quad n \geq 0, \tag{4}$$

where λ is a general Lagrangian multiplier (Inokuti *et al.*, 1978) which can be identified optimally via the variational theory, the subscript n denotes the n th-order approximation, \tilde{u}_n is considered as a restricted variation (He, 1999; 2000; He, *et al.*, 2004), *i.e.* $\delta \tilde{u}_n = 0$

In order to find the initial approximation, we present a new technique motivated by the standard Adomian decomposition method (SADM), take the initial approximation as

$$u_0 = u(x,0) + tu_x(x,0) + f(x) \tag{5}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$.

Also by following the idea in modification of Adomian decomposition method (MADM) we suggest another modification, that the initial approximation u_0 defined above in (5) be decomposed into two parts, namely $u_{0,1}$ and $u_{0,2}$ such that

$$u_0 = u_{0,1} + u_{0,2} \tag{6}$$

The proper choice of the parts $u_{0,1}$ and $u_{0,2}$ depends mainly on trial basis.

Analysis of Klein-Gordon Equation:

In this section, we present the solution of Eq. (1) subject to initial conditions (2) by means of VIM. First we construct a correction functional,

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) [(u_n)_{ss} - (\tilde{u}_n)_{xx} + bu_n + g(\tilde{u}_n) - f] ds, \quad n \geq 0, \tag{7}$$

where \tilde{u}_n is considered as restricted variations, which mean $\delta \tilde{u}_n = 0$. To find the optimal (s) , we proceed as follows:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) [(u_n)_{ss} - (\tilde{u}_n)_{xx} + bu_n + g(\tilde{u}_n) - f] ds, \tag{8}$$

and consequently

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(s) [(u_n)_{ss} + bu_{nn}] ds, \tag{9}$$

which results

$$\delta u_{n+1}(x, t) = \delta u_n(x, t)(1 - \lambda'(s)) + \delta(u_n)_s \lambda(s) + \int_0^t \delta u_n(x, s)(\lambda''(s) + b\lambda(s)) ds = 0. \tag{10}$$

The stationary conditions can be obtained as follows:

$$1 - \lambda'(s) = 0 \Big|_{s=t}, \quad \lambda(s) = 0 \Big|_{s=t}, \quad \lambda''(s) + b\lambda(s) = 0 \Big|_{s=t}, \tag{11}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda(s) = \frac{-\cos(\sqrt{bt}) \sin(\sqrt{bs}) + \sin(\sqrt{bt}) \cos(\sqrt{bs})}{\sqrt{b}}, \tag{12}$$

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left(\frac{-\cos(\sqrt{bt}) \sin(\sqrt{bs}) + \sin(\sqrt{bt}) \cos(\sqrt{bs})}{\sqrt{b}} \right) \times [(u_n)_{ss} - (u_n)_{xx} + bu_n + g(u_n) - f] ds. \tag{13}$$

Applications:

Example 1:

Consider the linear Klein-Gordon equation (El-Sayed, 2003)

$$u_{tt} - u_{xx} = u, \tag{14}$$

with the following initial conditions:

$$u(x,0) = 1 + \sin(x), \quad u_t(x,0) = 0 \tag{15}$$

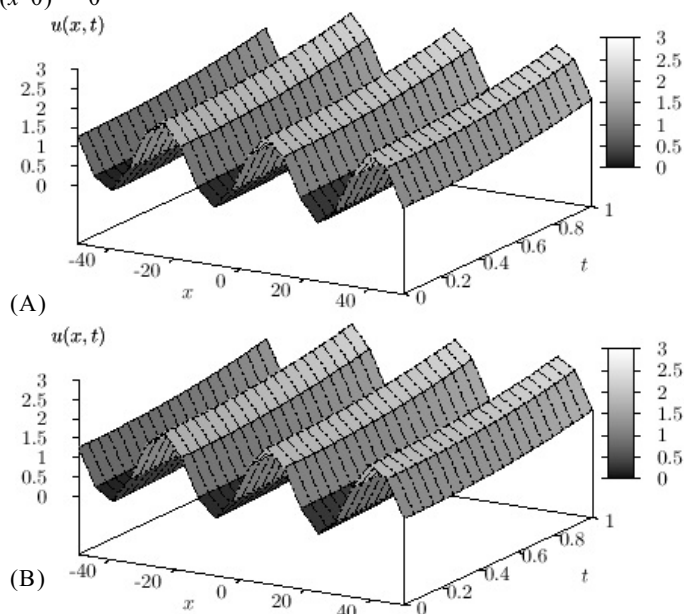


Fig. 1: (a) The exact solution (b) The numerical results for $u(x,t)$ by means of 2-iterate VIM solution, whose exact solution was found to be:

$$u(x,t) = \sin(x) + \cosh(t). \tag{16}$$

We can take the initial approximation in the form

$$u_0 = 1 + \sin(x). \tag{17}$$

The next iterate is easily obtained from Eq. (13) where $b = -1$ and is given by:

$$u_1 = \sin(x) + \frac{1}{2}e^t + \frac{1}{2}e^{-t}, \tag{18}$$

Fig. 1 shows the comparisons between the 2-iterate of VIM and the exact solution (16).

Example 2:

Consider the nonlinear Klein-Gordon equation (Wazwaz, 2006)

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \tag{19}$$

subject to the following initial conditions:

$$u(x,0) = 0 \quad u_t(x,0) = 0, \tag{20}$$

whose exact solution was found to be:

$$u(x,t) = x^3t^3. \tag{21}$$

The classical way to choose the initial approximation is to take $u_0 = u(x,0) = 0$. The first iterate is easily obtained from (13) where $b = 0$ and is given by:

$$u_1 = \frac{1}{56}x^6t^8 - \frac{3}{10}xt^5 + x^3t^3. \tag{22}$$

(23)

Now, by using the new technique which we presented in Eq. (5) we have

$$g(x) = 6xt(x^2 - t^2) + x^6t^6, \tag{24}$$

so that $f(x)$ which is arising from integrating the Eq.(24) is

$$f(x) = x^3t^3 - \frac{3}{10}xt^5 + \frac{1}{56}x^6t^8. \tag{25}$$

so, we have

$$u(x,0) = u(x,0) + tu_t(x,0) + f(x), \tag{26}$$

$$= x^3t^3 - \frac{3}{10}xt^5 + \frac{1}{56}x^6t^8. \tag{27}$$

The first iterate is easily obtained from (13) where $b = 0$ and given by:

$$u_1 = x^3t^3 - \frac{x^{12}t^{18}}{959616} + \frac{x^7t^{15}}{19600} + \frac{x^9t^{13}}{364} + \frac{9}{100}x^2t^{12} - \frac{1}{28}x^9t^{13} - \frac{9x^2t^{12}}{1100} + \frac{53x^4t^{10}}{4200}. \tag{28}$$

By using Eq. (6) we can take the initial approximation as

$$u(x,0) = x^3 t^3. \tag{29}$$

The first iterate is easily obtained from (13) and given by:

$$u_1 = x^3 t^3, \tag{30}$$

which is the exact solution. Fig. 2 shows comparisons between the exact solution (21) and 1-iterate of VIM using the three different initial approximations namely $u_0 = 0$, Eq.(27) and Eq. (29).

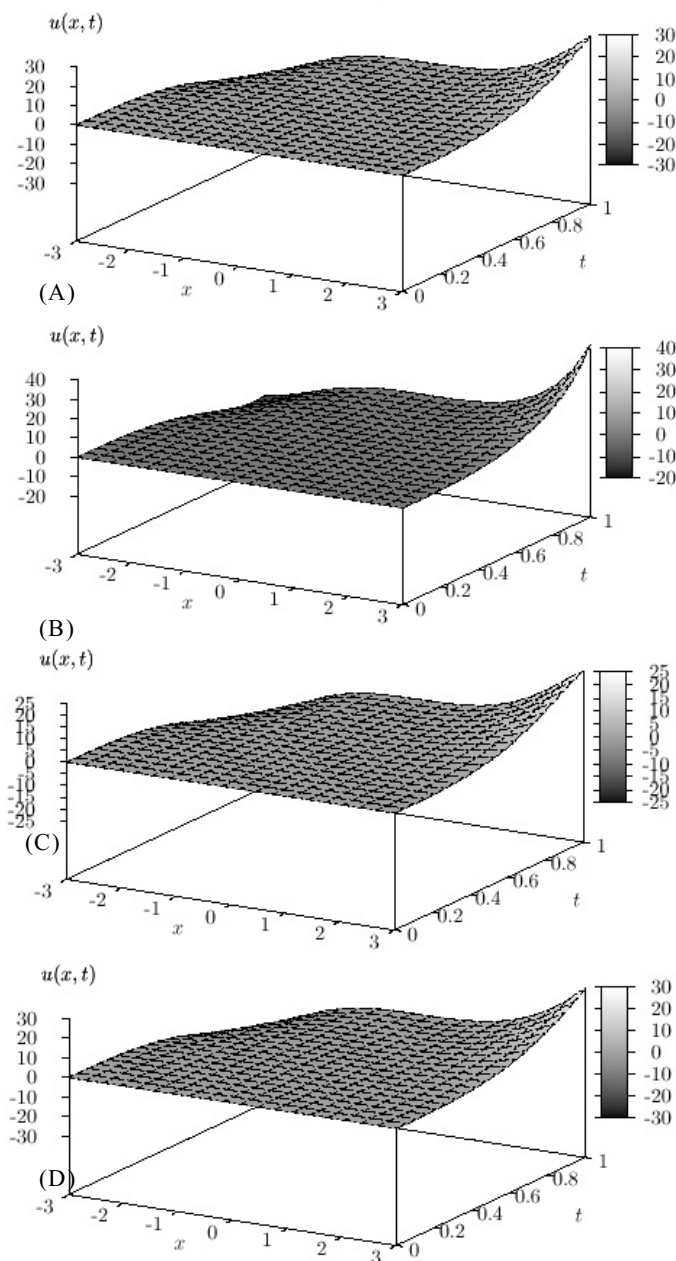


Fig. 2: (a) The exact solution. The numerical results for $u(x,t)$ by means of 1-iterate VIM solution for

$$u_0 = 0, (c) u_0 = x^3 t^3 - \frac{3}{10} x t^5 + \frac{1}{56} x^6 t^8 (d) u_0 = x^3 t^3$$

Example 3:

We next consider the nonlinear Klein-Gordon equation [11]

$$u_{tt} - u_{xx} + u^2 = -x \cos t + x^2 \cos^2 t, \tag{31}$$

subject to the following initial conditions:

$$u(x, 0) = x, \quad u_t(x, 0) = 0, \tag{32}$$

whose exact solution was found to be:

$$u(x, t) = x \cos t. \tag{33}$$

The classical way to choose the initial approximation is to take $u_0 = u(x, 0)$, so that in this example $u_0 = x$. The first iterate is easily obtained from Eq. (13) where $b = 0$ and is given by:

$$u_1 = \frac{x^2}{4} - \frac{x^2 t^2}{4} + x \cos^2 t - \frac{1}{4} x^2 \cos^2 t. \tag{34}$$

Now, by using the new technique which we presented in Eq. (5) we have

$$g(x) = -x \cos t + x^2 \cos t, \tag{35}$$

so, we have

$$u(x, 0) = x \cos t + \frac{1}{4} x^2 (t^2 - \cos t + 1). \tag{36}$$

The first iterate is easily obtained from Eq. (13) where $b = 0$ and given by:

$$\begin{aligned} u_1 = & x \cos t + \frac{1}{4} x^2 (t^2 - \cos t^2 + 1) - \frac{1}{256} x^4 + \frac{26}{9} x^3 \\ & + \frac{1}{8} t^2 \frac{1}{2} t^2 x^3 \cos t - 2 t x^3 \sin t - \frac{1}{32} t^2 x^4 \cos t^2 \\ & + \frac{1}{256} t^2 x^4 - \frac{1}{4} t^2 x^2 - \frac{1}{192} t^4 x^4 + \frac{1}{24} t^4 - \frac{1}{480} t^6 x^4 \\ & + \frac{1}{16} t x^4 \sin t \cos t - \frac{1}{4} x^2 \sin t^2 - \frac{17}{6} x^3 \cos t^3 \\ & - \frac{7}{2546} x^4 \cos t^4 - \frac{1}{8} \sin t^2 \end{aligned} \tag{37}$$

Now, by using Eq. (6) we can take the initial approximation as

$$u(x, 0) = x \cos t \tag{38}$$

The first iterate is easily obtained from Eq. (13) where $b = 0$ and given by:

$$u_1 = x \cos t,$$

which is the exact solution. Fig. 3 shows the comparisons between the exact solution (21) and 1-iterate of VIM using different initial approximations.

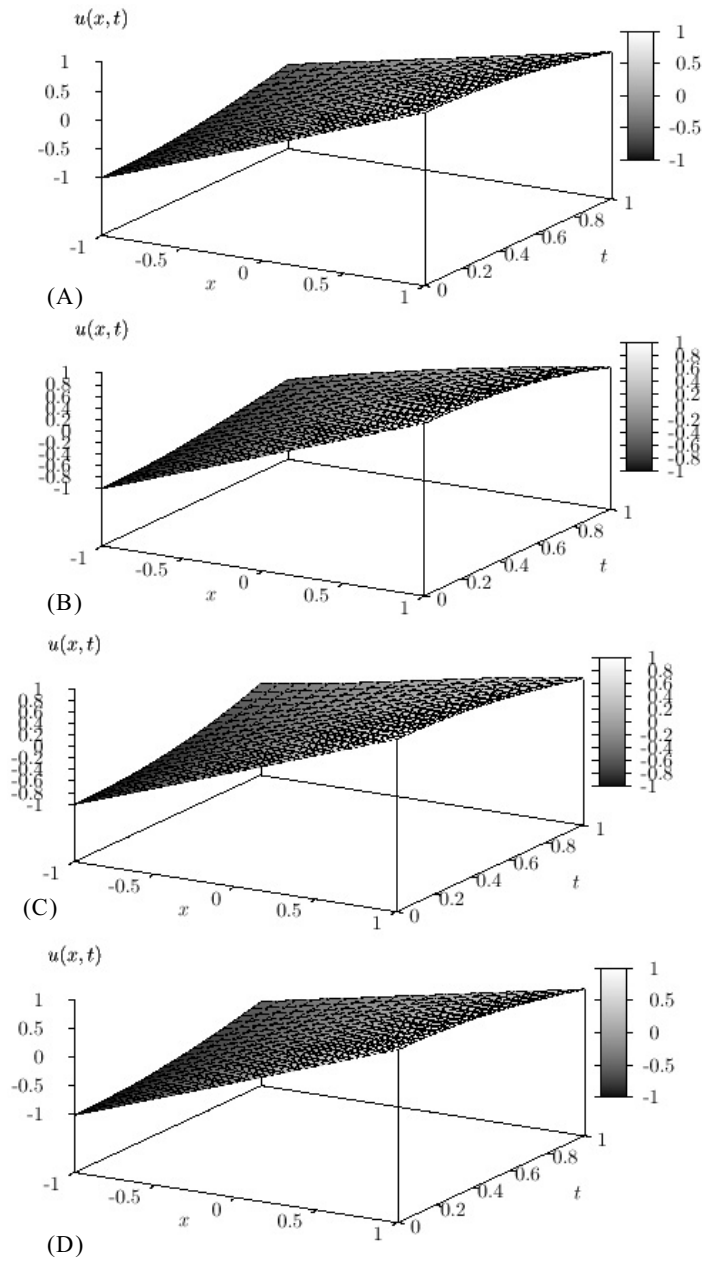


Fig. 3: (a)The exact solution. The numerical results for $u(x,t)$ by means of 1-iterate VIM solution for
 (b) $u_0 = \cos t + \frac{1}{4}x^2(t^2 - \cos^2 t + 1)$, (d) $u_0 = x \cos t$

Example 4:

Consider the nonlinear Klein-Gordon equation (Deeba and Khuri, 1996)

$$u_{tt} - u_{xx} - 2u = -2 \sin x \sin t, \tag{39}$$

with the following initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin x, \tag{40}$$

and exact solution:

$$u(x,t) = \sin x \sin t. \tag{41}$$

The classical way to choose the initial approximation is to take $u_0 = u(x,0)$ so that in this example $u_0 = 0$. The first iterate is easily obtained from Eq. (13) where $b = 2$ and given by:

$$u_1 = -\frac{1}{6} \sin x (\sqrt{2} e^{-(2\sqrt{2}t)} - \sqrt{2} - 4 \sin t e^{\sqrt{2}}) e^{(-\sqrt{2}t)}.$$

Again, by using the new technique which we presented in Eq. (5) we

$$u(x,0) = 2 \sin x \sin t - t \sin x. \tag{42}$$

The first iterate is easily obtained from (13) where $b = 2$ and given by:

$$u_1 = 2 \sin x \sin t - \sin x + \frac{1}{24} \sin x [-\sqrt{2} + \sqrt{2} e^{(2\sqrt{2}t)} - 16 \sin t e^{(\sqrt{2}t)} + 12 t e^{\sqrt{2}t}] e^{(-\sqrt{2}t)}.$$

Now, by using Eq. (6) we can take the initial approximation as

$$u(x,0) = t \sin x \tag{43}$$

The first iterate is easily obtained from Eq. (13) where $b = 2$ and is given by:

$$u_1 = t \sin x - \frac{1}{24} \sin x [\sqrt{2} e^{(2\sqrt{2}t)} - \sqrt{2} + 12 t e^{(\sqrt{2}t)} - 16 \sin t e^{(2t)}] e^{(-\sqrt{2}t)}.$$

Fig. 4 shows the comparisons between the exact solution (21) and 1-iterate of VIM using different initial approximations.

Example 5:

Consider the nonlinear Klein-Gordon equation of the form (Deeba and Khuri, 1996)

$$u_{tt} - u_{xx} + \frac{\pi^2}{4} u + u^2 = x^2 \sin \frac{\pi}{2} t, \tag{44}$$

subject to the following initial conditions:

$$u(x,0) = 0, \quad u_t(x,0) = \frac{\pi}{2} x, \tag{45}$$

and exact solution:

$$u(x,t) = x \sin \frac{\pi}{2} t. \tag{46}$$

A simple way to choose the initial approximation is to take $u_0 = u(x,0)$ so that in this example $u_0 = 0$. The first iterate is easily obtained from Eq. (13)

where $b = \frac{\pi^2}{4}$ and given by: 4

$$u_1 = -\frac{4}{3\pi^2} x^2 [2 \cos(m) - 3 \cos(m)^2 + \cos(m)^4 + \sin(m)^2 + \sin(m)^2 \cos(m)^2],$$

where $m = \frac{\pi t}{2}$.

As before, by using the new technique which we presented in Eq.(5) we have

$$u(x, t) = \frac{\pi x t}{2} - \frac{x^2}{2\pi^2} + \frac{1}{4} \left(t^2 + \frac{2 \cos \pi t}{\pi^2} \right) x^2. \tag{47}$$

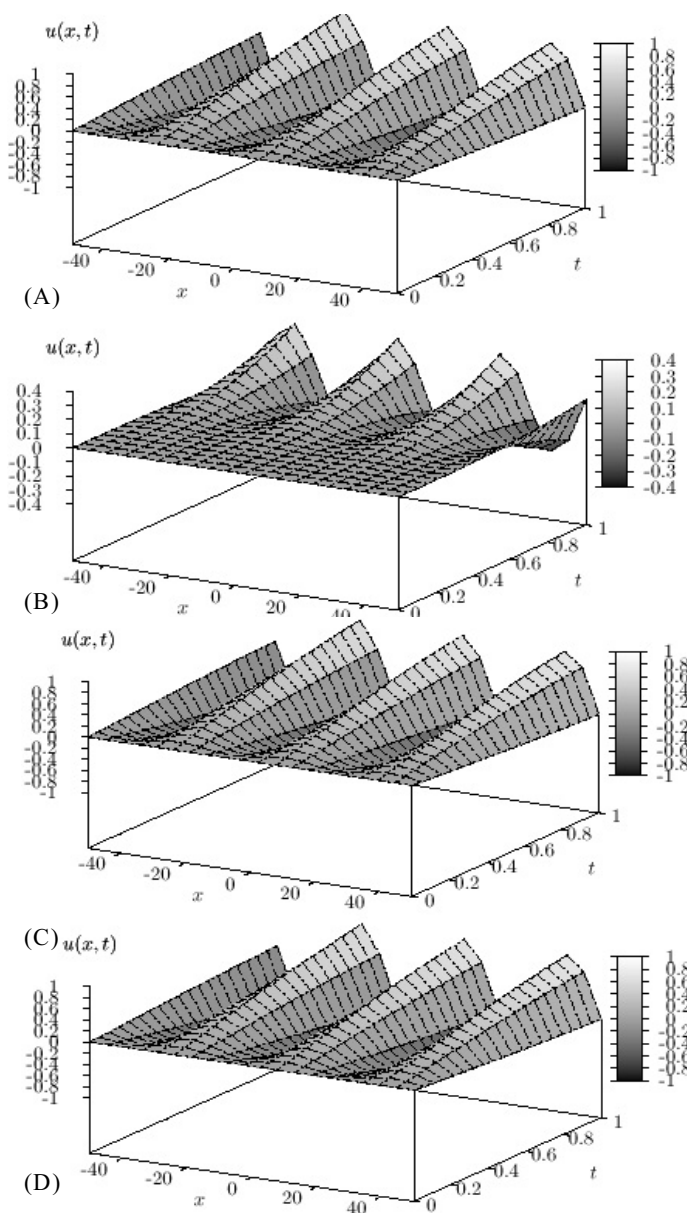


Fig. 4: (a) The exact solution. The numerical results for $u(x,t)$ by means of 1-iterate VIM solution for (b) $u_0 = 0$, (c) $u_0 = 2 \sin x \sin t - t \sin x$, (d) $u_0 = t \sin x$

The first iterate is easily obtained from (13) and given by:

$$\begin{aligned}
 u_1 = & \frac{\pi xt}{2} - \frac{x^2}{2\pi^2} + \frac{1}{4} \left(t^2 + \frac{2 \cos(\pi t)}{\pi^2} \right) x^2 - \frac{1}{960} (-1860x^4 + 120x^4x^4 - 960\pi^2) \\
 & - 480\pi^3x^3 + 25\pi^8x^2t^4 - 60\pi^6t^2x^2 + 1920x^4 \cos(m) - 60x^4 \cos(m)^2 \\
 & - 480\pi^3x^3 \cos(m) + 480\pi^4t^2 + 960tx^4 \sin(m) - 20\pi^4t^4x^4 - 80\pi^5t^3x^3 \\
 & - 120\pi^4x^2 \cos(m) + 20\pi^9t^3 - 240\pi^2t^2x^4 \cos \pi t + 2\pi^6x^4t^6 + 960\pi^2x^3 \sin(m) \\
 & - 40\pi^6t^4 + 180\pi^2t^2x^4 + 12\pi^7x^3t^5) / \pi^6 - 120\pi^4x^2 \cos(m) \\
 & + 20\pi^9t^3 - 240\pi^2t^2x^4 \cos \pi t + 2\pi^6x^4t^6 + 960\pi^2x^3 \sin(m)
 \end{aligned}$$

where $m = \frac{\pi t}{2}$.

As before, by using the new technique which we presented in Eq.(6) we have

$$u(x,0) = \frac{\pi xt}{2}. \tag{48}$$

The first iterate is easily obtained from (13) and given by:

$$\begin{aligned}
 u_1 = & \frac{1}{\pi^2} \left\{ \frac{\pi tx}{2} - \frac{1}{6} x [-6 \sin(m)\pi^2 + 64 \cos(m)x + 8 \cos(m)^4 x \right. \\
 & + 3 \cos(m)^2 \pi^3 t + 8 \sin(m)^2 - 27 \cos(m)^2 x + 3 \sin(m)^2 \pi^3 t \\
 & \left. + 6x \cos(m)^2 \pi^3 t^2 - 56 \sin(m)^2 x + 6x \sin(m)^2 \pi^2 t^2 \right\}.
 \end{aligned} \tag{49}$$

where $m = \frac{\pi t}{2}$.

Fig. 5 shows the comparisons between the exact solution (46) and 1-iterate of VIM using different initial approximations.

Example 6:

Finally, we consider the homogeneous Klein-Gordon nonlinear hyperbolic equation

$$u_{tt} - \alpha u_{xx} + \beta u + \gamma u^3 = 0, \tag{50}$$

subject to the following initial conditions:

$$u(x,0) = B \tan(Kx), u_t(x,0) = BcK \sec^2(Kx), \tag{51}$$

and exact solution:

$$u(x,t) = B \tan(K(x+ct)), \tag{52}$$

where c, α, β and γ are constants, and

$$B = \sqrt{\frac{\beta}{\gamma}}, K = \sqrt{\frac{-\beta}{2\alpha + 2c^2}}. \tag{53}$$

The classical way to choose the initial approximation is to take $u_0 = u(x,0)$ so that in this example $u_0 = B \tan(Kx)$. The first iterate is easily obtained from Eq. (13) where $b = \beta$ and given by:

$$\begin{aligned}
 u_1 = & B \tan(Kx) + B \sin(Kx)(2\alpha K^2 \cos(\sqrt{\beta t}) + B^2 \gamma \cos(\sqrt{\beta t})) \\
 & + \beta \cos(Kx)^2 \cos(\sqrt{\beta t}) - B^2 \gamma \cos(\sqrt{\beta t}) \cos(kx)^2 \\
 & - 2\alpha K^2 \cos(\sqrt{\beta t})^2 + B^2 \gamma \sin(\sqrt{\beta t})^2 \cos(kx)^2 \\
 & - 2\alpha K^2 \sin(\sqrt{\beta t})^2 - \beta \cos(kx)^2 \cos(\sqrt{\beta t})^2 \\
 & - \beta \cos(Kx)^2 \sin(\sqrt{\beta t})^2 - B\gamma \cos(\sqrt{\beta t})^2 \\
 & + B^2 \gamma \cos(\sqrt{\beta t})^2 \cos(Kx)^2 - B\gamma \cos(\sqrt{\beta t})^2 / \cos(Kx)^3 \beta.
 \end{aligned}
 \tag{54}$$

As before, by using Eq.(5) we have

$$u(x,0) = B \tan(Kx) + BcKt \sec(Kx).
 \tag{55}$$

The first iterate is easily obtained from (13) and given by:

$$\begin{aligned}
 u_1 = & B \tan(Kx) + BcKt \sec(Kx)^2 - B(3\beta^{(11/2)})B^2 \cos(\sqrt{\beta t})^2 \gamma \sin(Kx)c^2 K^2 t^2 \cos(Kx) \\
 & + 3\beta^{(11/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma \sin(Kx)c^2 K^2 t^2 \cos(Kx) \\
 & - 3\beta^5 B^2 \sin(\sqrt{\beta t})\gamma cK \cos(Kx)^2 - \beta^{(11/2)}B^2 \cos(\sqrt{\beta t})^2 \gamma \sin(Kx) \cos(Kx)^5 \\
 & + \beta^{(11/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma \sin(Kx) \cos(Kx)^3 - 2\beta^{(11/2)} \cos(\sqrt{\beta t})\alpha K^2 \sin(Kx) \cos(Kx)^3 \\
 & - \beta^{(11/2)}B^2 \cos(\sqrt{\beta t}) \gamma \sin(Kx) \cos(Kx)^3 + 3\beta^{(6/2)}B^2 \cos(\sqrt{\beta t}) \gamma \sin(Kx)c^2 K^2 \cos(Kx) \\
 & + \beta^{(11/2)}B^2 \cos(\sqrt{\beta t}) \gamma \sin(Kx) \cos(Kx)^5 - 6\beta^{(9/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma c^3 K^3 t \\
 & - 3\beta^{(11/2)}B^2 \cos(\sqrt{\beta t})\gamma cKt \cos(Kx)^4 - \beta^{(11/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma \sin(Kx) \cos(Kx)^5 \\
 & + 6\beta^4 B^2 \sin(\sqrt{\beta t})\gamma c^3 K^3 + 3\beta^5 B^2 \sin(\sqrt{\beta t})\gamma cK \cos(Kx)^4 \\
 & - 6\beta^{(9/2)}B^2 \cos(\sqrt{\beta t})^2 \gamma c^3 K^3 t + \beta^{(11/2)}B^2 \cos(\sqrt{\beta t})^2 \gamma \sin(Kx) \cos(Kx)^3 \\
 & + 2\beta^{(11/2)} \cos(\sqrt{\beta t})^2 \alpha K^2 t \sin(Kx) \cos(Kx)^3 + 3\beta^{(11/2)}B^2 \cos(\sqrt{\beta t})^2 \gamma cKt \cos(Kx)^2 \\
 & + 2\beta^{(11/2)} \sin(\sqrt{\beta t})^2 \alpha K^2 t \sin(Kx) \cos(Kx)^3 + 3\beta^{(13/2)} \sin(\sqrt{\beta t})^2 \sin(Kx) \cos(Kx)^5 \\
 & - 6\beta^5 \sin(\sqrt{\beta t})\alpha K^3 c \cos(Kx)^2 - \beta^6 \sin(\sqrt{\beta t})cK \cos(Kx)^4 \\
 & + 3\beta^{(11/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma cKt \cos(Kx)^2 - 3\beta^{(11/2)}B^2 \sin(\sqrt{\beta t})^2 \gamma cKt \cos(Kx)^4 \\
 & - \beta^{(13/2)} \cos(\sqrt{\beta t}) \sin(Kx) \cos(Kx)^5 + 4\beta^5 \sin(\sqrt{\beta t})\alpha K^3 c \cos(Kx)^4 \\
 & + \beta^{(13/2)} \cos(\sqrt{\beta t})^2 \sin(Kx) \cos(Kx)^5 + \beta^{(11/2)}B^2 \cos(\sqrt{\beta t})^2 \gamma c^3 K^3 t^3 \\
 & + \beta^{(13/2)} \sin(\sqrt{\beta t})^2 cKt \cos(Kx)^4 + \beta^{(13/2)} \cos(\sqrt{\beta t})^2 cKt \cos(Kx)^4 \\
 & + 6\beta^{(11/2)} \cos(\sqrt{\beta t})^2 \alpha K^3 ct \cos(Kx)^2 - 4\beta^{(11/2)} \sin(\sqrt{\beta t})^2 \alpha K^3 ct \cos(Kx)^4
 \end{aligned}$$

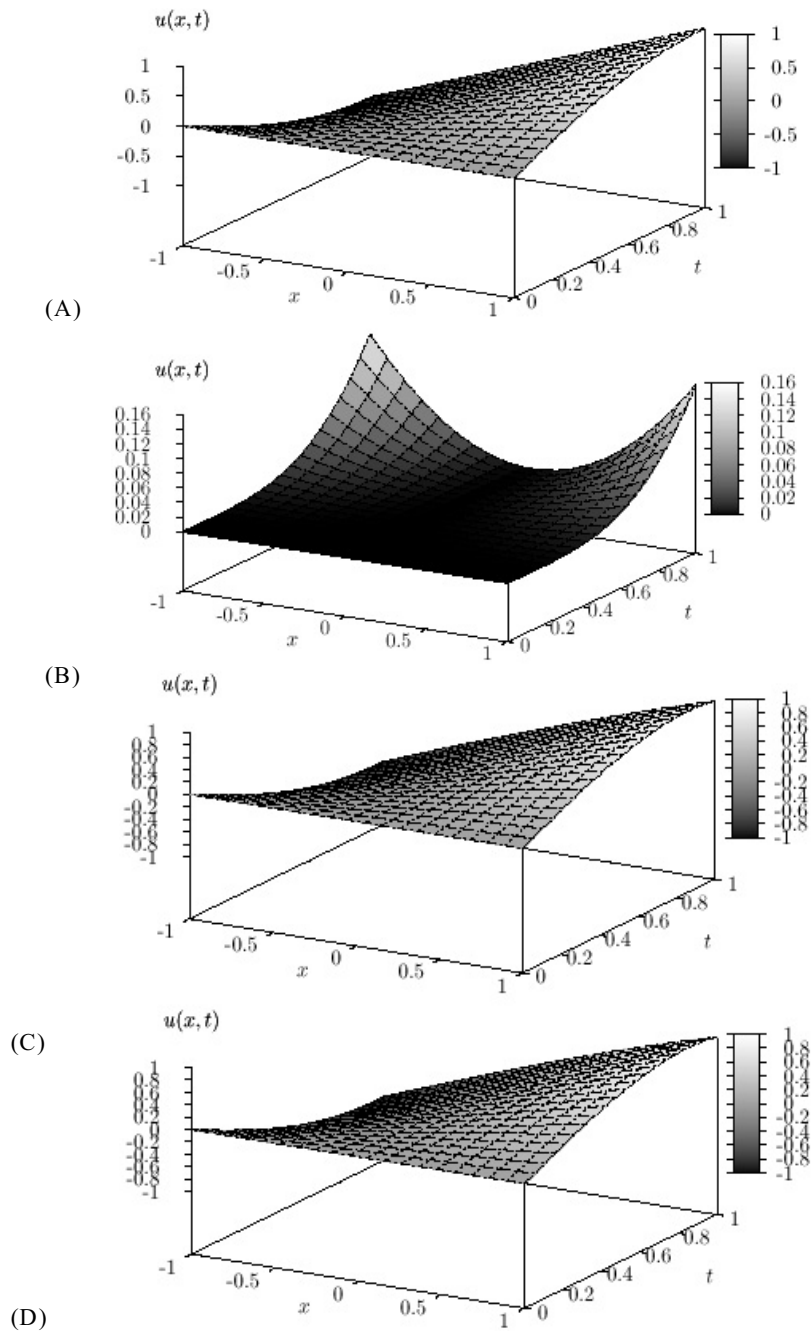


Fig. 5: (a) The exact solution. The numerical results for $u(x,t)$ by means of 1-iterate VIM solution for

$$(b)u_0 = 0, (c)u_0 = \frac{\pi xt}{2} - \frac{x^2}{2\pi^2} + \frac{1}{4}\left(t^2 + \frac{2 \cos \pi t}{\pi^2}\right)x^2 (d)u_0 = \frac{\pi xt}{2}$$

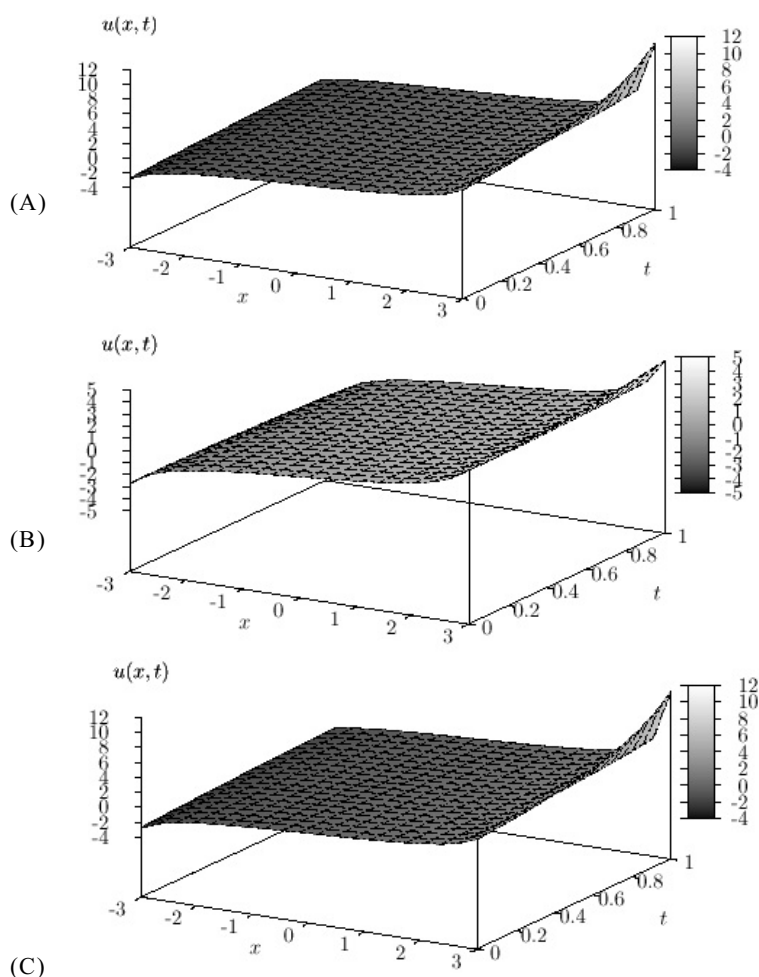


Fig. 6: (a) The exact solution. The numerical results for $u(x,t)$ by means of 1-iterate VIM solution for

(b) $u_0 = B \tan(Kx)$, (c) $u_0 = B \tan(Kx) + BcKt \sec^2(Kx)$, where $B = 0.816497$, $K = 0.426401$, $c = 0.5$, $\alpha = 2.5$, $\beta = -1.0$, $\gamma = -1.5$

$$\begin{aligned}
 &+6 \beta^{(11/2)} \sin(\sqrt{\beta t})^2 \alpha K^3 ct \cos(Kx)^2 \\
 &-6 \beta^{(9/2)} B^2 \cos(\sqrt{\beta t})^2 \gamma \sin(Kx) c^2 K^2 \cos(Kx) \\
 &-4 \beta^{(11/2)} \cos(\sqrt{\beta t})^2 \alpha K^3 ct \cos(Kx)^4 - 6 \beta^{(9/2)} B^2 \sin(\sqrt{\beta t})^2 \gamma \sin(Kx) c^2 K^2 \cos(Kx) \\
 &+ \beta^{(11/2)} B^2 \sin(\sqrt{\beta t})^2 \gamma c_3 K^3 t^3 / \cos(Kx)^6 \beta^{(13/2)}.
 \end{aligned}$$

Fig. 6 shows the comparisons between the exact solution (52) and 1-iterate of VIM using different initial approximations.

Conclusions:

In this paper, the variation iteration method (VIM) has been successfully employed to obtain the approximate analytical solutions of the Klein-Gordon equation. The new technique for choosing initial approximation has been showed effectively and more rapid convergent series solution. Comparisons with the exact solution reveal that VIM is very effective and convenient. It is shown that VIM is a promising tool for nonlinear partial differential equation.

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