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Quasi-likelihood Estimation of the Parameters of the Weibull Extension Model

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Abstract: This article is concerned with maximum quasi-likelihood estimation of the unknown parameters of the Weibull extension model. A numerical comparison between the maximum likelihood and the quasi-likelihood estimates will be carried out and show that the efficiency of the quasi-likelihood estimations relative to the maximum likelihood is relatively higher. It has been seen that the obtained estimators are not available in a nice closed forms, although they can be easily evaluated numerically.

Keywords:

INTRODUCTION

Recently, Tang *et al.,* (2003). Suggested a new failure model called Weibull extension model, which is asymptotically related to the traditional two-parameter Weibull distribution that can be used to model bathtub-shaped failure rate . The probability density function of the Weibull extension model (WEM) is

f(t) =l b (t /a)b-1 exp [(t /a) b +la (1– e (t/a) b)] (1)
, a , b, l > 0 , t ³ 0 ,

where a , b and l are parameters ,the cumulative distribution and failure rate of (1) are

F(t) = 1– exp [- la (e (t/a) b-1)], (2)

and

H(t) = lb (t /a)b-1 exp [(t /a)b], (3)

respectively .

Tang *et.al.,* (2003) discussed the properties of WEM, investigated its characteristics and applying the model to real lifetime data . They discussed its statistical inferences and hypothesis tests using maximum likelihood method.
The kth non-central moment of (1) is given by

$$\mu'_k = E\,(T^k) = \eta\ \alpha^k\ e^{\eta}\ J^k, \qquad k=1,\ 2,\ \ldots, \qquad (4)$$

where $\eta = \alpha\lambda$, $J_k = \int_1^{\infty} (\log w)^{1/\beta}\ e^{-\eta w}\ dw$, and $w = e^{(t/\alpha)^{\beta}}$

the central moments of (1) can be derived as

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$$\mu'_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(1-i)} \mu'_1 \ \mu_1'^{(k-i)} \ , \ k=1, 2, \ldots\ldots \qquad (5)$$

When $\beta \leq 1$, the failure rate function (3) is an increasing function , and if $0<\beta<1$, the failure rate function has bathtub – Shape property (Xie *et al.,* (2002)). The change point of bathtub curve t* and the corresponding minimal failure rates H (t*) at the change point (Tang *et al.,* (2003)) are:

t* = a (1/b - 1) 1/b and

H (t*) = lb (1/b - 1)1-1/b e1/b-1 (6)

The quasi-likelihood function was introduced by Wedderburn (1974) for estimating the unknown parameters in generalized linear models when only the variance of each observation is specified to be either equal to, or proportional to some function of its expectation . He defined the quasi-likelihood , strictly the quasi-log likelihood, Q for an observation t with mean m and variance V (μ) by the equation

$$Q\ (t\ ;\ \mu) = \int^{\mu} \frac{T-u}{V(u)} du \qquad (7)$$

Plus some function of t only , or equivalently by

$$\frac{\partial Q(t\ ;\mu)}{\partial \mu} = (\ T - \mu) \ /\ V\ (\mu) \qquad (8)$$

where $\mu = E$ (T), V (μ) = var (T). The variance assumption is generalized to var (T) = φ V(μ), where the variance function V(.) is assumed to be known and the parameter φ may be unknown .The quasi-likelihood function has properties similar to those of the log likelihood function (Wedderburn (1974), McCullagh (1983)). If the underlying distribution comes from a natural exponential family the quasi-likelihood maximize the likelihood and so have full asymptotic efficiency as the maximum likelihood estimates (Firsth (1987)); under more general distribution there is some loss of efficiency, Firsth (1987) discussed the asymptotic relative efficiency.

In this article, we derive the quasi-likelihood function and the maximum quasi-likelihood estimators (MQLEs)for the WEM parameters and a numerical comparison is carried out and show that the efficiency of the quasi-likelihood estimate, relative to maximum likelihood (ML), is high. In section 2, the maximum likelihood function, and maximum likelihood estimates (MLE) of the unknown parameters of the WEM are derived. Section 3 deals with the quasi-likelihood function and maximum quasi-likelihood estimates (MQLE) of the unknown parameters of the WEM. A numerical example is used as illustration in section 4.

The Maximum Likelihood Estimates:

Suppose that $t_1, \ldots, t_n$ is a sample of size n have a WEM (1). The likelihood function (LF) is given by

$$L(\lambda,\alpha,\beta) = \lambda^n \beta^n \prod_{i=1}^{n} Z_i^{\beta-1} \exp\{\sum_{i=1}^{n} Z_i^{\beta} + \sum_{i=1}^{n} \alpha\lambda (1- e^{Z_i^{\beta}})\} \qquad (9)$$

where $Z_i = t_i/\alpha$. The derivatives of the natural logarithm of likelihood function with respect to λ , α and β will be

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n\alpha - \alpha \sum_{i}^{n} Z_i^{\beta}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log Z_i + \sum_{i=1}^{n} Z_i^{\beta} - \log Zi - \lambda\alpha \sum_{i=1}^{n} e^{Z_i^{\beta}} Z_i^{\beta} \log Z_i^{\beta} ,$$

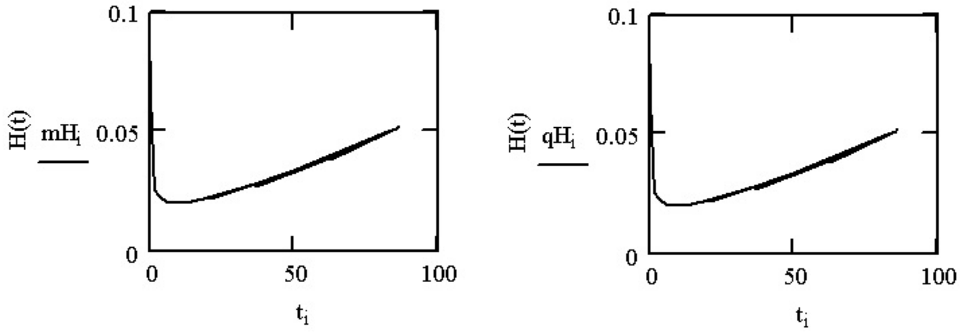

Fig. 1: Plot of the failure rate function (3) with mle (mHi)and mqle(qHi) for Aarset data .

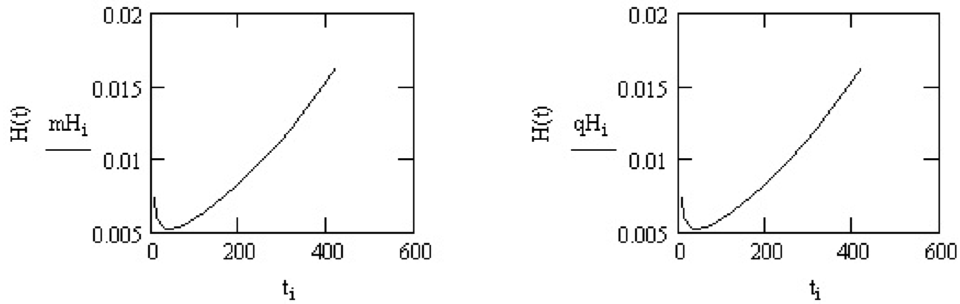

Fig. 2: plot of the failure rate function (3) with mle (mHi)and mqle(qHi) for Wang data .

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$$\frac{\partial \log L}{\partial \alpha} = n\lambda + \frac{n\,(\beta-1)}{\alpha} - \frac{\beta}{\alpha}\sum_{i=1}^{n} Z_i^{\beta} - \lambda \sum_{i=1}^{n} (1 - \beta\, Z_i^{\beta})\, e^{Z_i^{\beta}} \qquad (10)$$

Equating equations (10) by zero , the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ of α , β , λ will be the solution of the following equations :

$$\sum_{i=1}^{n} e^{C_i} - n - \frac{n}{\hat{\alpha}\hat{\lambda}} = 0 \ ,$$

$$n + \sum_{i=1}^{n} (C_i + 1)\log C_i - \hat{\alpha}\hat{\lambda} \sum_{i=1}^{n} C_i\, e^{C_i} \log C_i = 0 \ ,$$

and

$$n + \sum_{i=1}^{n} (C_i) - \hat{\alpha}\hat{\lambda} \sum_{i=1}^{n} C_i\, e^{C_i} = 0 \qquad (11)$$

where $C_i = Z_i^{\hat{\beta}}$, $i = 1 , 2, \dots , n$.

Numerical solution of equations and computer facilities are used to obtain $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ As the scale parameter a approaches infinity, the WEM (2) has Weibull distribution with shape parameter b as a special case (Tang *et.al.,* (2003). The element of the observed information matrix

$$Iij\,(\theta) = E\left(-\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right) \qquad i, j = 1, 2, 3, \ldots$$

Where q = (α, β, λ) are as follows:

$$\left.\frac{-\partial^2 \log L}{\partial\alpha^2}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{-n(\beta-1)}{\alpha^2} - \frac{\beta(\beta+1)}{\alpha^2}\sum_{i=1}^{n} C_i - \frac{\lambda\beta}{\alpha}\sum_{i=1}^{n} C_i e^{C_i}(1-\beta(C_i+1))$$

$$\left.\frac{-\partial^2 \log L}{\partial\beta^2}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{-1}{\beta^2}\left[-n+\sum_{i=1}^{n} C_i \log^2 C_i - \alpha\lambda\sum_{i=1}^{n} C_i e^{C_i}(C_i+1)\log^2 C_i\right]$$

$$\left.\frac{-\partial^2 \log L}{\partial\lambda^2}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{n}{\lambda^2}$$

$$\left.\frac{-\partial^2 \log L}{\partial\alpha\partial\beta}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{n}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n} C_i(1+\log C_i) - \lambda\sum_{i=1}^{n} C_i e^{C_i}\left[1+\beta\left(C_i+1-\frac{1}{\beta}\right)\right]\log C_i$$

$$\left.\frac{-\partial^2 \log L}{\partial\alpha\partial\lambda}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = -n + \sum_{i=1}^{n}(1-\beta C_i)e^{C_i}$$

$$\left.\frac{-\partial^2 \log L}{\partial\beta\partial\lambda}\right|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{\alpha}{\beta}\sum_{i=1}^{n} C_i e^{C_i}\log C_i \qquad (12)$$

$I_{ij}(\theta)$ were obtained by Tang *et.al.,* (2003) in the case of random censored sample.

By usual large sample normal approximation and under mild conditions, the joint distribution of $\hat\alpha, \hat\beta, \hat\lambda$ is asymptotically normal with mean q and covariance matrix $I^{-1}(\theta) = (I_{ij}(\theta))^{-1}$.

The Quasi-maximum Likelihood Estimates:

Now, derive the MQLEs for the unknown parameters of the WEM, consider (4), then the mean and variance of WEM are

$$\mu = \mu_1' = \eta\alpha e^{\eta} J_1$$

$$Var(t) = \mu^2\left[\frac{e^{-\eta}J_2}{\eta J_1^2} - 1\right] = V(\mu)^2 \qquad (13)$$

Where

$$\varphi = \left[\frac{e^{-\eta}J_2}{\eta J_1^2} - 1\right] = V(\mu)^2 = \mu^2$$

and

$$J_i = \int_1^{\infty} (\log w)^{\frac{i}{\beta}} e^{-\alpha\lambda w} \, dw \qquad , \qquad i = 1,\ 2$$

For a random sample of size n and using (7) the quasi – likelihood function of the WEM (1) is given by

$$\frac{\partial Q(t\,;\mu)}{\partial \mu} = \frac{\sum_{i=1}^{n} t_i - n\mu}{\mu^2} \qquad (14)$$

Then

$$Q(t\ ;\mu) = -\,D\ /\ \mu - n \log \mu. \qquad (15)$$

Where $D = \sum_{i=1}^{n} t_i$

By substituting for m as a function of α, β, λ ,.

$$\mu = \lambda\alpha^2 e^{\alpha\lambda} \int_1^{\infty} (\log w)^{\frac{1}{\beta}} e^{\alpha\lambda w} \, dw = \lambda\alpha^2 e^{\alpha\lambda} J_1$$

we get $$Q(t, \alpha, \beta, \lambda) = \frac{De^{-\alpha\lambda}}{\alpha^2\lambda J_1} - n[\log(\alpha^2\lambda J_1) + \alpha\lambda] \qquad (16)$$

Differentiating (16) with respect to a , b and l we get .

$$\frac{\partial Q}{\partial \alpha} = \frac{D[2J_1 + \alpha\lambda D1]\ e^{\lambda\alpha}}{\alpha^3 \lambda J_1^2} - n[\frac{\lambda D1}{J_1} + \frac{2}{\alpha}]$$

$$\frac{\partial Q}{\partial \alpha} = \frac{I_2}{\beta^2 J_1} - n[\frac{De^{-\alpha\lambda}}{\alpha^2\lambda J_1}] \qquad (17)$$

$$\frac{\partial Q}{\partial \lambda} = \frac{-\,De^{-\alpha\lambda}[\ J_1 + \alpha\lambda\ D1\ ]}{\alpha^2\lambda^2 J_1^2} - n[\frac{\lambda D1}{J_1} + \frac{1}{\lambda}]$$

Where $D_1 = J_1 - I_1$

$$I_1 = \int_1^{\infty} w(\log w)^{\frac{1}{\beta}} e^{-\alpha\lambda w} \, dw$$

,

$$I_2 = \int_1^{\infty} (\log(\log w))^{\frac{1}{\beta}} e^{-\alpha\lambda w} \, dw$$

Equating derivatives (17) to zero, and solving for $\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*$ the quasi-likelihood estimators can be obtained. Again numerical technique of computer facilities are needed to obtain $\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*$.

Table 1: MLE and MQLE for Aarset data

parameter	MLE	MQLE
α	3.99	3.501
β	0.30	0.30
λ	0.001	0.0023
log L (a , b, l)	- 377.98	- 334.92

Table 2: MLE and MQLE for Wang data

parameter	MLE	MQLE
α	3.171	3.75
β	0.016	0.008
λ	0.201	0.238
log L (a , b, l)	- 169.382	- 184.01

Table 3: MLES and MQLES of α , β and λ for a = 1.5 , b = 0.5, and n = 10, 20 , 30 , 50 , 100 for simulated data.

n	parameter	MLE	MQLE	SD(MLE)	SD(MQLE)	MSE(MLE)	MSE(MQLE)
10	α	1.606	1.215	0.177	0.091	0.043	0.089
	β	0.370	0.49	0.122	2.88x10-14	0.032	1x10-4
	λ	0.00033	0.00039	3.77x10-5	1.878x10-5	4.451x10-7	5.1x10-7
20	α	1.727	1.230	0.187	0.067	0.087	0.078
	β	0.49	0.49	9.79x10-3	0.0021	2.039 x10-4	5.68 x10-4
	λ	0.00036	0.00028	4.085x10-5	1.4x10-5	4.104 x10-7	5.18 x10-7
30	α	1.811	1.24	0.16	0.058	0.122	0.0073
	β	0.49	0.489	2.88x10-14	2. 088x10-14	1 x10-4	1.85 x10-15
	λl	0.0004	0.0003	3.58x10-5	1.511x10-5	$3.88x10^{-5}$	$1.511x10^{-5}$
50	α	1.88	1.25	0.145	0.058	0.166	0.118
	β	0.49	0.489	2.88x10-14	5.86x10-15	1x10-4	4x10-5
	λ	0.0005	0.0003	3.105x10-5	2.086x10-5	3.704x10-7	2.08x10-5
100	α	1.96	1.496	0.112	0.055	0.122	0.096
	β	0.498	0.493	2.208x10-15	1.212x10-15	1x10-5	1.879x10-6
	λ	0.0008	0.00095	2.33x10-5	2.135x10-5	3.5x10-6	4.94x10-7

The element of the quasi- observed information matrix

$$QI_{ij}(\theta) = E\left(-\frac{\partial^2 Q(t;\alpha,\beta,\lambda)}{\partial\theta_i\,\partial\theta_j}\right), \qquad i, j = 1, 2, 3, \ldots$$

$$\left.\frac{-\partial^2 \log Q}{\partial\alpha^2}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = \frac{-De^{-\alpha\lambda}}{\alpha^4 J_1^3}\left(\alpha J_1 D5 - D6\left(\frac{3}{\lambda} - 2\alpha I_1\right)\right) - \frac{n}{J_1^2\alpha^2}\left(\lambda^2\alpha^2 D4 - 2J_1^2\right)$$

$$\left.\frac{-\partial^2 \log Q}{\partial\beta^2}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = \frac{-n}{\beta^4 J_1^2}\left(I_2\left(I_2 - 2\beta J_1 I_3\right)\right) - \frac{2De^{-\alpha\lambda}}{\alpha^2\beta^2\lambda J_1^3}\left(-I_2 J_1\beta\right)$$

$$\left.\frac{-\partial^2 \log Q}{\partial\lambda^2}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = -\frac{n}{J_1^2\lambda^2}\left(\alpha^2\lambda^2 D4 + J_1^2\right)$$

$$+ \frac{De^{-\alpha\lambda}}{\alpha^2\lambda^3 J_1^3}\left[\alpha\lambda J_1\left(\frac{-J_1}{\alpha} + I_1 - D1 + \alpha\lambda\left(D1 + D2\right)\right) + 2\left(J_1 + \alpha\lambda D1\right)\left(J_1 - \alpha\lambda I_1\right)\right]$$

$$\left.\frac{-\partial^2 \log Q}{\partial\alpha\partial\beta}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = \frac{De^{-\alpha\lambda}}{\alpha^3\beta^2\lambda J_1^3}[-2J_1I_2 + \alpha\lambda(J_1D3 - 2I_2D1)] - \frac{n\lambda}{\beta^2 J_1^2}(J_1D3 - D1I_2)$$

$$\left.\frac{-\partial^2 \log Q}{\partial\alpha\partial\lambda}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = -\frac{De^{-\alpha\lambda}}{\alpha^3\lambda^2 J_1^3}[\alpha\lambda J_1D5 - D6(J_1 - 2\alpha\lambda I_1)] + \frac{n}{J_1^2}(J_1D1 - \alpha\lambda D4)$$

$$\left.\frac{-\partial^2 \log Q}{\partial\beta\partial\lambda}\right|_{\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*} = \frac{n\alpha D7}{\beta^2 J_1} - \frac{De^{-\alpha\lambda}}{\alpha\beta^2\lambda^2 J_1^3}[\lambda J_1(I_4 + I_2) - I_2(J_1 + 2\lambda I_1)] \qquad (18)$$

where $D2 = I_1 - I_3$, $D3 = I_2 - I_4$, $D4 = J_1D2 - I_1 D1$,

$D5 = D1 - 2(I_1 + J_1) - \alpha\lambda(D1+D2)$, $D6 = 2J_1 + \alpha\lambda D1$, $D7 = J_1 I_4 - I_1 I_2$,

$$I_3 = \int_1^\infty w^2 (\log w)^{\frac{1}{\beta}} e^{-\alpha\lambda w}\, dw, \quad I_4 = \int_1^\infty w(\log(\log w))(\log w)^{\frac{1}{\beta}} e^{-\alpha\lambda w}\, dw$$

and

$$I_5 = \int_1^\infty (\log(\log w))^2 (\log w)^{\frac{1}{\beta}} e^{-\alpha\lambda w}\, dw$$

A Numerical Examples:

To illustrate the usefulness of the proposed estimators of quasi-likelihood method, we considered here two real data sets as follows: (i) first initially reported by Aarset (1987) to identify the bathtub hazard rate contains life time of 50 devices. Hence we obtained the proposed estimators for Aarset (1987) data and summarized it in Table (1) . The MATHCAD (2001) program is used to evaluate these estimates . The inverse of the observed information matrix for MLEs (Var-Cov($\hat{\theta}$)) and MQLEs (Var-Cov ($\hat{\theta}^*$)) are:

$$\text{Var-Cov}((\hat{\theta}) = \begin{pmatrix} 9.8\times10^{-2} & 0.017 & -1.9\times10^{-5} \\ & 2.7\times10^{-3} & -3.4\times10^{-5} \\ & & 1.2\times10^{-6} \end{pmatrix}$$

$$\text{Var-Cov}(\hat{\theta}^*) = \begin{pmatrix} 2.04\times10^{-3} & 2.87\times10^{-13} & 3.4\times10^{-15} \\ & 1.48\times10^{4} & -1.01\times10^{-8} \\ & & 4.61\times10^{-6} \end{pmatrix}$$

(ii) The second real data for the life time to failure of 18 of an electronic device used by Wang (2000), the results are summarized in Table (2). The inverse of the observed information matrix MLEs (Var-Cov($\hat{\theta}$)) and MQLEs (Var-Cov($\hat{\theta}^*$)) are:

$$\text{Var-Cov}\ (\hat{\theta}) = \begin{pmatrix} 1.298 & -1.317\times10^{-4} & 0.096 \\ & 1.423\times10^{-5} & -1.04\times10^{-5} \\ & & 4.87\times10^{-6} \end{pmatrix}$$

$$\text{Var-Cov}\ (\hat{\theta}^{*} = \begin{pmatrix} 0.09 & -3.031\times10^{-12} & -8.96\times10^{-10} \\ & 5.46\times10^{-6} & 2.24\times10^{-6} \\ & & 3.81\times10^{-9} \end{pmatrix}$$

The MATHCAD (2001) program is used to generate five independent sets of M=5000 samples of sizes 10 , 20 , 50 and 100 from a WEM With $\alpha = 1.5$, $\beta = 0.5$ and $\lambda = 0.001$. For each sample, the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ the quasi- likelihood estimates $\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*$ of α , β and λ are obtained and there variances and mean square errors calculated empirically.

Estimates , variances and mean square errors are displayed in Table (3) , results shows that the standard deviations and the mean square errors of the MQLE of α , β and λ are less than the corresponding MLE for all sample sizes, that is, the results indicate that the MQLES are more efficient than the corresponding MLES for all sample sizes .Future numerical results using different values for α , β and λ are needed to establish strong conclusion .

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