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Right Perfect Right Self-Injective Rings

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Abstract: There is a conjecture due to Faith which asserts that a left (or right) perfect, right self-injective ring is quasi-forbenius, here we prove this conjecture without any extra condition. 2000 Mathematics Subject Classification 16D50, 16L60

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INTRODUCTION

As a property of the self injective ring we prove that every finitely generated right nonregular module is semisimple if R is right self injective. Also we prove that every right perfect right self injective ring is right artinian and hence quasi- frobenius

A ring R is called quasi-Frobenius if R is noetherian on one side and is injective or cogenerator on one side.

A well known result of Osofsky asserts that a left perfect, left and right self-injective ring is quasi-Frobenius, and the Faith conjecture asserts that a left (or right) perfect, right self-injective ring is quasi-forbenius. Here we prove this conjecture.

Throughout this paper all rings considered are associative with unity and all modules are unitary R-module. We write M_R to indicate a right R-modules, Rad R for the Jacobson Radical and $A_R << B_R$ to means that A is a small submodule of B_R.A ring R is called right simple injective if for every ideal I of R and every R-module homomorphism $g : I \to R$ with simple image then $\gamma = c.$ is a left multiplication by an element c of R.

The following lemmas will be needed in our investigation.

Lemma 1 (2.7[4]):

Every finitely cogenerated torsionless right R-module embeds in a free module R^n of finite rank.

Lemma 2 [3]:

Let R be an arbitrary ring

If $0 \to A \to M \to W \to 0$ is a split exact sequence of right R-modules then M is reflexive if and only if A and W are reflexive.

Lemma 3 (9.3.4[3]):

If R is a ring such that R /Rad R is semisimple then we have:

Every simple right respectively left R-module is isomorphic to a submodule of $(R/\text{Rad } R)_R$ respectively $_R(R/\text{Rad } R)$.

Lemma 4(12.1.1 [1]):

The following are equivalent for a ring R

- Every finitely generated R-module is reflexive
- $_R R$ and R_R are cogenerators
- R_R is a cogenerator and $_R R$ is injective
- $_R R$ is a cogenerator and R_R is injective
- $_R R$ and R_R are injective and to every simple R-module there is an isomorphic ideal in R.

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Definition 1:

A ring which satisfies the conditions of the above lemma is called a ring with perfect duality

Lemma 5(12.1.2 [1]):

If R is a ring with perfect duality then R is semiperfect and both ${}_R R$ and R_R are finitely cogenerated.

Definition 2[3]:

A ring with perfect duality is quasi-frobenius if and only if it is noetherian on one side (see 13.2.2[3]). Here we construct a proposition which is true for every finitely generated module M which is not regular i.e. M does not equal R_R for any ring R.

Proposition 1:

If R is a right (or left) self-injective ring then every finitely generated right (or left) R-module M (not equal R_R) is semisimple.

Proof:

We prove the proposition in case of right R-module and the proof of left R-module can be done by a similar way.

Since R_R is right self-injective so it is right simple injective.

Step1:

Since R is right simple injective ring then for any ideal $I \subseteq R$, cI is simple, for some element $c \in R$. Write cI to be of the form $a_i R$. The epimorphism $\alpha_i : R \to a_i R$ will induce the epimorphism
$\alpha: \oplus_n R \to \oplus_n a_i R$. $\alpha(a_i) = (a_i r_i)$and ke $\alpha = \{(a_i) s.t. (a_i I) = 0\}$, Being $a_i R$ simple implies that $\oplus_n a_i R$ is semisimple.

Step2:

Let M be a finitely generated module with generators $\{m_i, i=1, 2 \ldots n\}$ so there is an epimorphism β and a free module F such that
$\beta : F \to M$. Since each finitely generated module is a factor of a free module with finite rank so
$\beta : F \cong \oplus_n R \to M \cong \oplus_n R/K$; $K = \ker\beta$, is an epimorphism given by
$\beta(x_i) = \sum m_i (x_i)$. On other hand $m_i = (r_i) + K$
So $\beta(x_i) = \sum[(r_i) + K](x_i)$.
$\operatorname{Ker} \beta = \{(x_i) \in \oplus_n R; \sum m_i (x_i) = 0\}$
$= \{(x_i) \in \oplus_n R; (r_i)(x_i) \in K\}$.
If $(y_i) \in \ker\alpha$, where α is as in step (1), then $(a_i)(y_i) = 0$ for some
$(a_i) \in \oplus_n R$. By which we get
$0 = \beta(0) = \beta(a_i y_i) = \sum [(r_i) + K](a_i y_i) = \sum [(r_i a_i) + K](y_i)$ i.e.
$(y_i) \in \ker \beta$, hence $\ker \alpha \subseteq \ker \beta$ and β induce an epimorphism

$$\bar{\beta} : \oplus_n a_i R \cong \oplus_n R/\ker\alpha \to M$$

Hence M is epimorphic image of a semisimple module by which M is semisimple and finitely generated so it is finitely cogenerated.

Notice:

Note that we used here the fact that the epimorphic image of a semisimple module is semisimple, but the epimorphic image of a semisimple ring need not be semisimple.

The ring $R = Z_8$ is self injective ring and the right R module $M = 2Z_8$ is cyclic R module but it is not semisimple this is because M is regula module ($2Z_8$ is a ring).

Note that in the book of Kasch (Kash F., 1982) it is mentioned that the epimorphic image of a semisimple ring is semisimple because he assumed from the beginning that every ring homomorphism is unitary, but this is not true in general.

Proposition (2):

If R is right perfect right self injective ring then every simple right R- module is torsionless and R is a ring with perfect duality.

Proof:

(i) Since R is right perfect then R/ Rad R is semisimple. By lemma (2) each simple module T is isomorphic to a submodule of $(R / \text{Rad } R)_R$, B, say.

Now since R is right perfect then the radical of any free right module F is small in F. Let F be of finite rank, i.e., $F \cong \oplus_n R$ so Rad F = 0 (F is finitely generated) by which $F \cong F/ \text{Rad } F \cong \oplus_n R$. Since there is a monomorphism $\varphi : R / \text{Rad } R \to \oplus_n (R/ \text{Rad } R) \cong \oplus_n R$ hence R / Rad R is torsionless as the free module $F \cong \oplus_n R$ is torsionless so the submodule B of R / Rad R is therefore torsionless by which every simple right R module T is torsionless.

(ii) Since every finitely generated right R-module M is semisimple so $M = T_1 \oplus T_2 \oplus \ldots \oplus T_n$, where T_i is simple and torsionless so M is torsionless.

Since M is torsionless and finitely cogenerated then by lemma (1) M can be embedded in a free module K with finite rank, so we can get the split exact sequence

$$0 \longrightarrow \text{Im} \cong M \longrightarrow K \longrightarrow K/M \longrightarrow 0$$

Im ψ is the image of ψ (K is semisimple so Im ψ is a direct summand of K). By lemma (2) M is reflexive as K is reflexive (also every ring with identity is reflexive) and so R is a ring with perfect duality.

Theorem 1:

Every right perfect right self injective ring is right artinian

Proof:

(a) As R_R is self injective, then by proposition (1) above every finitely generated right R- module is semisimple this means that R/I (I is a right ideal of R which does not equal R_R) is semisimple and finitely generated so it is finitely cogenerated this imply that every factor module R/I of R is finitely cogenerated for every ideal $I \neq R_R$.

(b) Since R is a ring with perfect duality so R_R and ${}_R R$ are finitely cogenerated (by lemma 5), hence $R_R/R_R \cong R_R$ is finitely cogenerated

By (a) and (b) every factor module R/I of R is finitely cogenerated and R is right artinian

As an application of the above study we prove the Faith conjecture which asserts that a left (or right) perfect, right self-injective ring is quasi-forbenius

Theorem 2:

Every right perfect right self injective ring is quasi frobenius.

Proof:

Since R_R is right artinian, so it is right noetherian. Now R is a ring with perfect duality and right noetherian so it is quasi frobenius (definition 2)

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